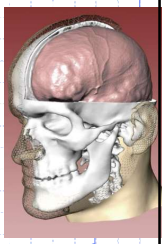
ME8026: Additive Manufacturing for Medical Applications

Tessellation Algorithms



Instructor: Dr. Prashant K. Jain
Professor (ME Discipline)
PDPM Indian Institute of Information Technology, Design and Manufacturing Jabalpur, Jabalpur, INDIA
Resources: web.iiitdmj.ac.in/~pkjain/
Email: pkjain@iiitdmj.ac.in, pkjain2006@gmail.com
http://www.iiitdmj.ac.in/Faculty/pkjain.html
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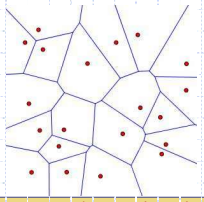
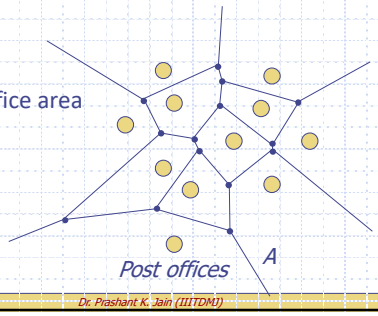
What is Voronoi Diagram

- A Voronoi diagram is a subdivision of a plane into regions based on distance to points in a specific subset of the plane.
- It is named after Georgy Voronoi, and is also called a **Voronoi tessellation**, a **Voronoi decomposition**, a **Voronoi partition**, or a **Dirichlet tessellation**
- The set of points also called seeds, sites, or generators



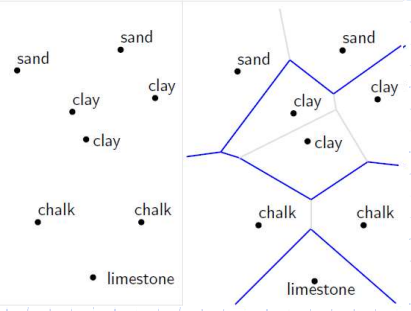
Applications

Post Office: What is the area of service?



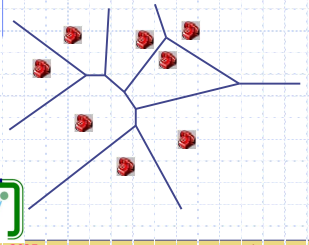
Applications

- Suppose we tested the soil at a number of sample points and classified the results



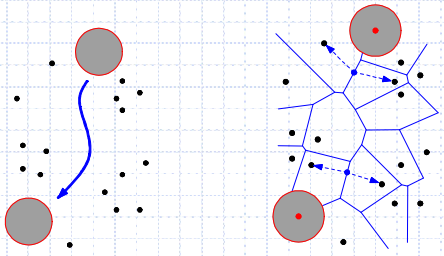
Applications

- Find nearest hospital, restaurant, gas station, telephone booth etc.



Applications

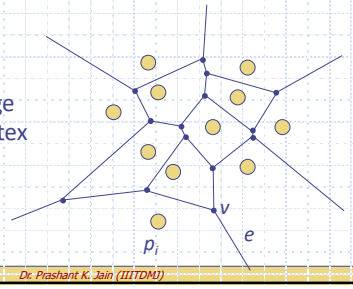
- Robot motion planning



Mathematical Definition

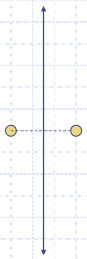
- Voronoi diagram is the partitioning of a plane with points into convex polygons such that each polygon contains exactly one generating point and every point in a given polygon is closer to its generating point than to any other

p_i : site points
e : Voronoi edge
v : Voronoi vertex



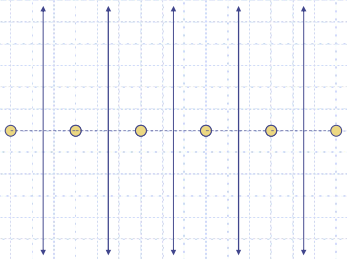
Voronoi Diagram Example: 1 site

Two sites form a perpendicular bisector

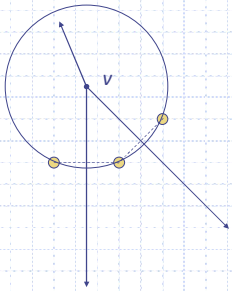


Voronoi Diagram is a line that extends infinitely in both directions, and the two half planes on either side.

Collinear sites form a series of parallel lines

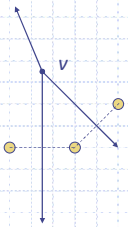


Non-collinear sites form Voronoi half lines that meet at a vertex

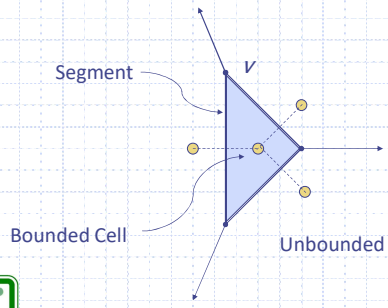


A Voronoi vertex is the center of an empty circle touching 3 or more sites.

Voronoi Cells and Segments

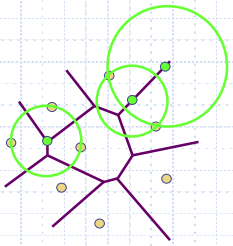


Voronoi Cells and Segments



Voronoi Diagram Properties

- A vertex of a Voronoi diagram is the center of a circle passing through three sites.
- Each point on an edge is the center of a circle passing through two sites.



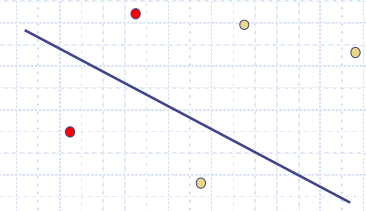
Implementations

- Incremental Algorithm
- Divide and Conquer Algorithm
- Fortune's Algorithm (Sweep line)

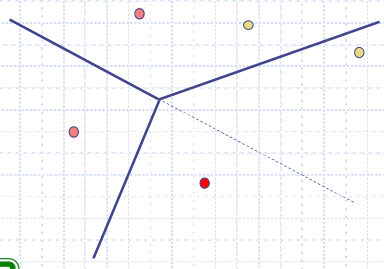
Incremental Algorithm

- Starting with any point of $\{p_1 \ldots . p_i\}$ add point p_{i+1}
- Explore all candidates to find the site p_j closest to p_{i+1}
- Draw the perpendicular bisector between the selected points
- Select closest points again and again and draw the perpendicular bisector until the last point of the given data set is taken into consideration

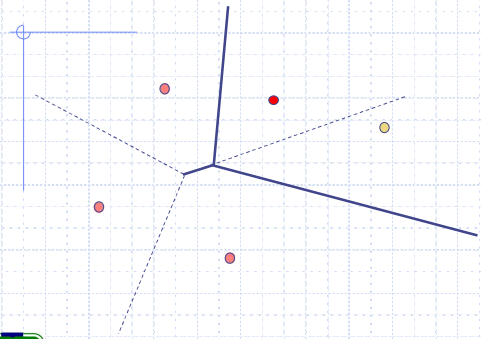
Example



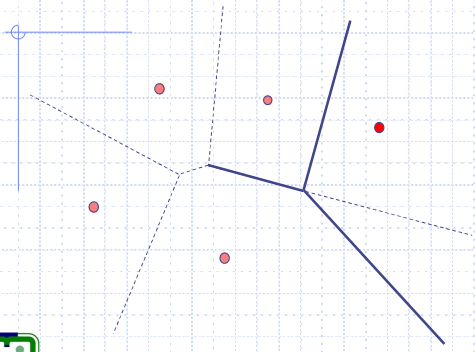
Example



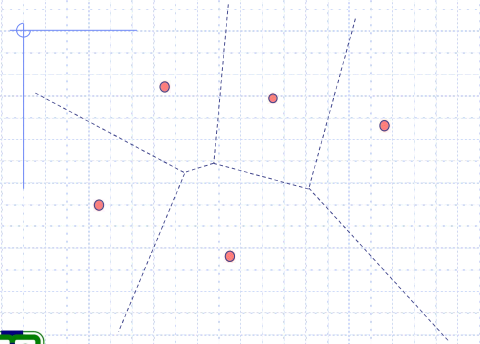
Example



Example



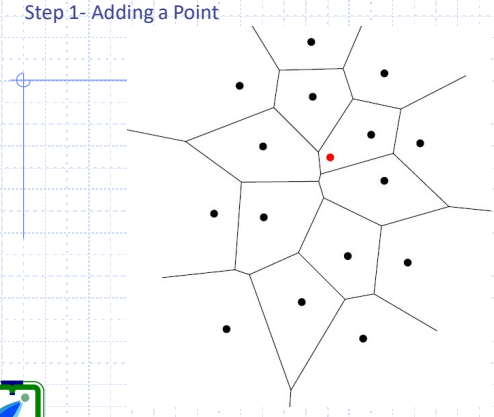
Example



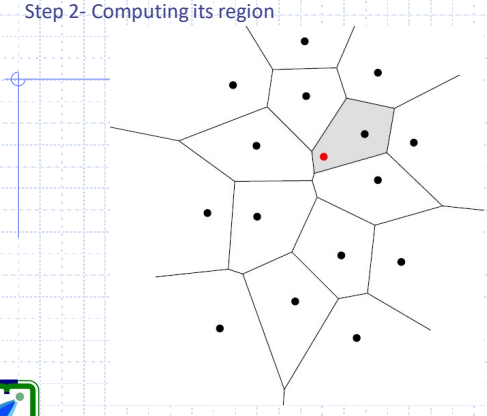
Modification in the Voronoi diagram

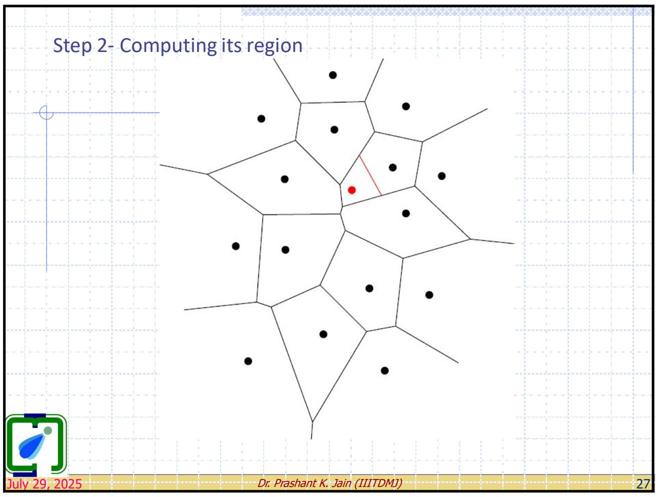
- A single point is inserted at a time in the Voronoi Diagram
- Computing the region
- Draw perpendicular bisector between new points and its neighbour points

Step 1- Adding a Point



Step 2- Computing its region




Step 2- Computing its region
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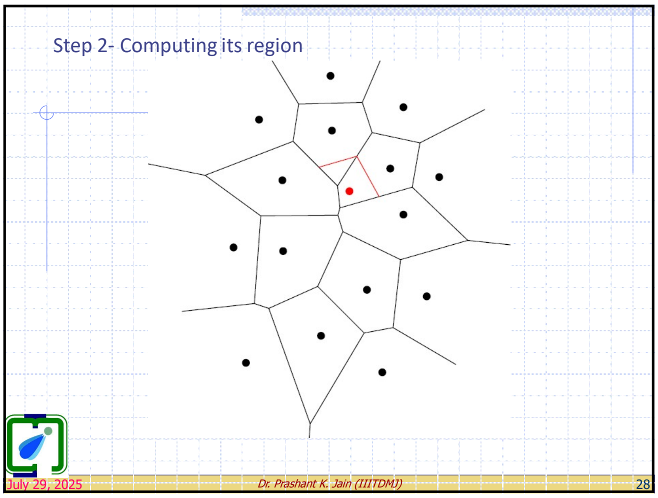
Step 2- Computing its region
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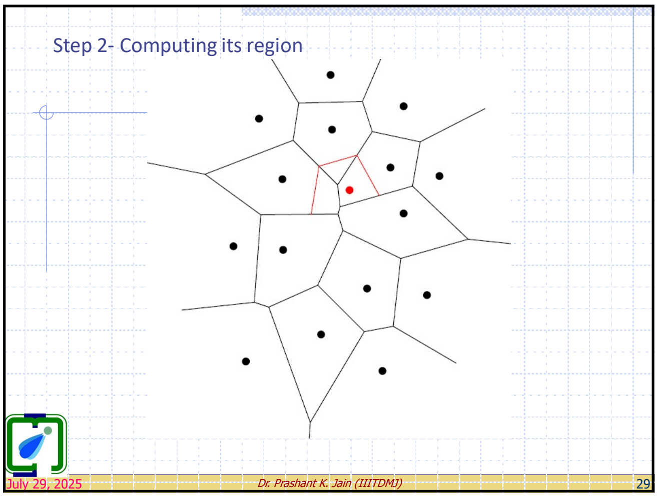
Step 2- Computing its region
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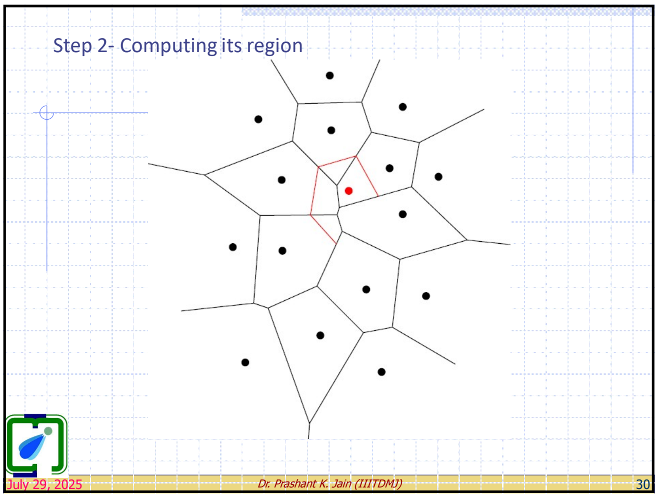
Step 2- Computing its region
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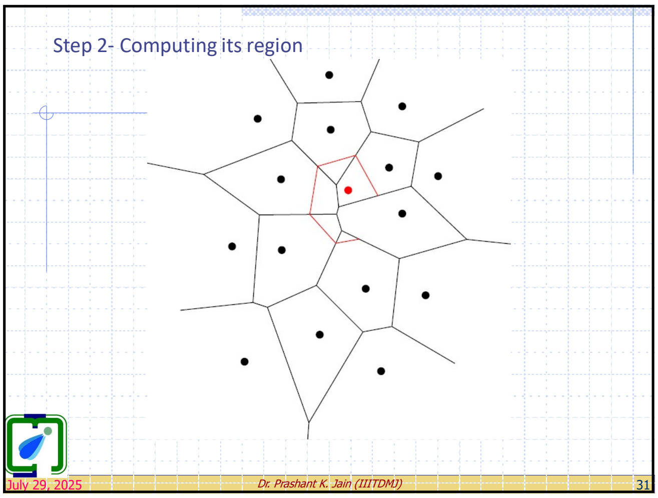
Step 2- Computing its region
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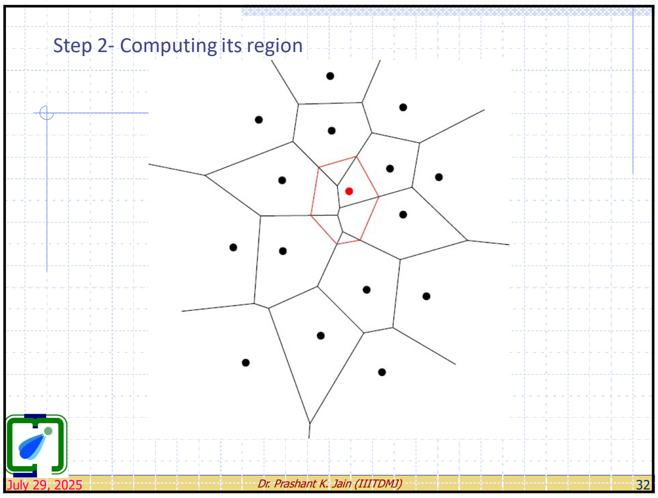
Step 2- Computing its region
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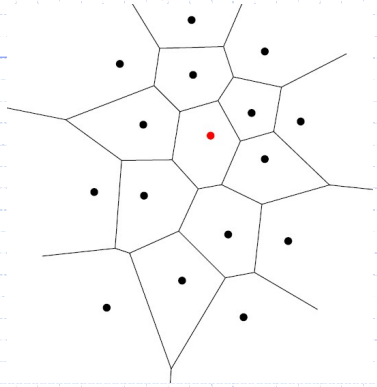
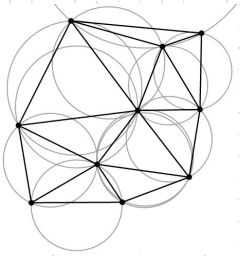
Step 3- Prune the initial diagram



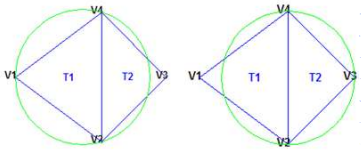
Delaunay Triangulation

- A Delaunay Triangulation (DT) for a set P of points in a plane is a triangulation DT(P) such that no point in P is inside the circumcircle of any triangle in DT(P).
- The triangulation is named after Boris Delaunay for his work on this topic from 1934



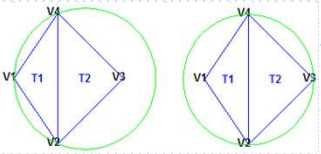
Contd..

- Circumcircle associated with each triangle contains no other point in its interior.
- The circumcircle associated with T1 is empty. It does not contain a point in its interior.
- The circumcircle associated with T2 is empty. It does not contain a point in its interior.
- This triangulation is a Delaunay triangulation



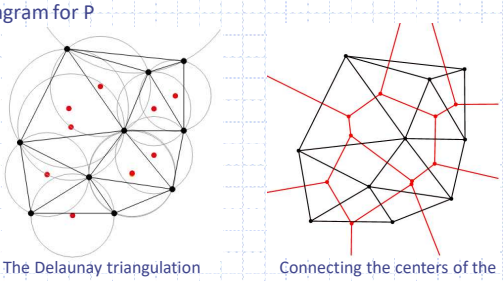
Contd..

- The triangles below are different.
- The circumcircle associated with T1 is not empty. It contains V3 in its interior.
- The circumcircle associated with T2 is not empty. It contains V1 in its interior.
- This triangulation is not a Delaunay triangulation.



Relationship with the Voronoi diagram

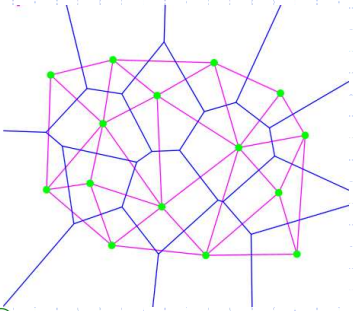
- The Delaunay triangulation of a discrete point set P in general position corresponds to the dual graph of the Voronoi diagram for P



The Delaunay triangulation with all the circumcircles and their centers (in red).

Connecting the centers of the circumcircles produces the Voronoi diagram (in red).

Dual Nature

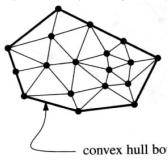


Voronoi Diagram
Delauny Triangulation

B. Delaunay: *Sur la sphère vide, Izvestia Akademii Nauk SSSR, Otdelenie Matematicheskikh i Estestvennykh Nauk, 7:793–800, 1934*

Delaunay Triangulation properties

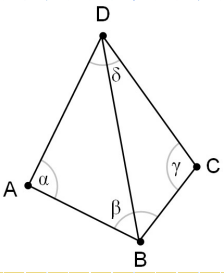
- Delaunay triangulation is a straight-line dual graph of the Voronoi Diagram of a point set.
- Maximize the minimum angle of all the angles of the triangles in the triangulation.
- A circle circumscribing any Delaunay triangle does not contain any other input points in its interior. Circumcircles of all the triangles are empty.
- Outer polygon must be convex hull



B. Delaunay: *Sur la sphère vide, Izvestia Akademii Nauk SSSR, Otdelenie Matematicheskikh i Estestvennykh Nauk, 7:793–800, 1934*

Flipping

- Looking at two triangles ABD and BCD with the common edge BD .
- If the sum of the angles α and γ is less than or equal to 180°, the triangles meet the Delaunay condition.
- This is an important property because it allows the use of a flipping technique.



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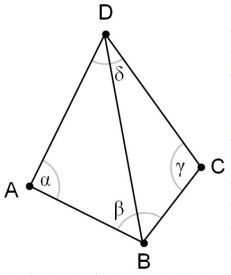
- Triangles with large internal angles are selected over ones with small internal angles.
- The triangles in the non-Delaunay triangulation have sharp angles at vertices B and D.
- If the edge {B, D} were replaced by an edge joining A and C, the minimum angle would be maximized and the triangulation would become a Delaunay triangulation.



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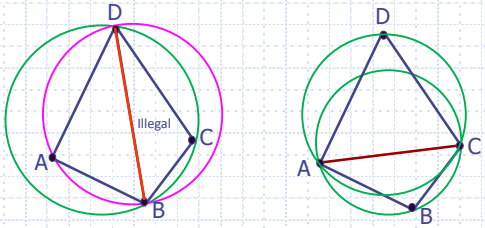
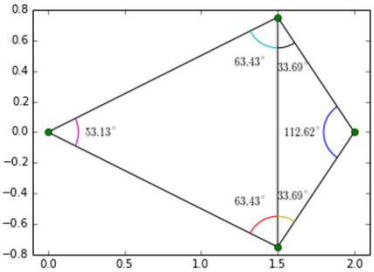
- If two triangles do not meet the Delaunay condition, switching the common edge BD for the common edge AC produces two triangles that do meet the Delaunay condition



Flipping through animation

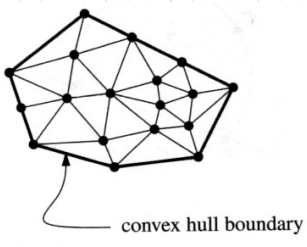
- If the connecting edge altered, the minimum angle would be maximized and the triangulation would become a Delaunay triangulation.



Source: https://en.wikipedia.org/wiki/Delaunay_triangulation

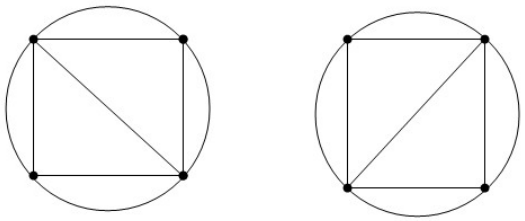
Triangulation Details

- For P consisting of n points, all triangulations contain 2n-2-k triangles, 3n-3-k edges
- n = number of points in P
- k = number of points on convex hull of P



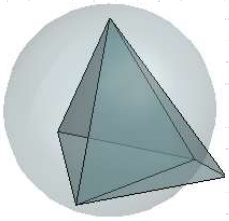
Non-unique Delaunay Triangulation

- In two dimensions, degeneracies arise when four or more unique points lie on the same circle.
- The vertices of a square, for example, have a non-unique Delaunay triangulation



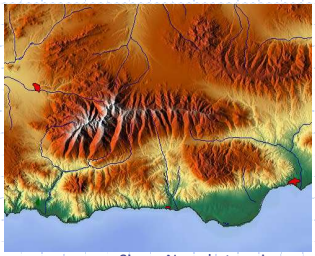
3D Delaunay triangulation

- 3-D Delaunay triangulation made up of two tetrahedra
- The circumsphere of one tetrahedron is shown to highlight the empty circumsphere criterion
- A 3-D Delaunay triangulation produces tetrahedra that satisfy the empty circumsphere criterion.



Applications

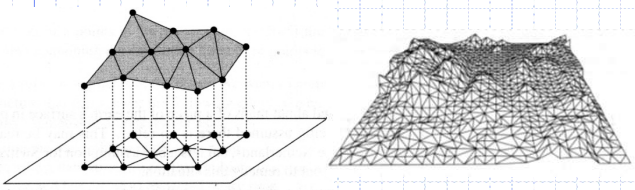
- For modelling terrain or other objects given a set of sample points, the Delaunay triangulation gives a nice set of triangles to use as polygons in the model. In particular, the Delaunay triangulation avoids narrow triangles.



Sierra Nevada terrain map

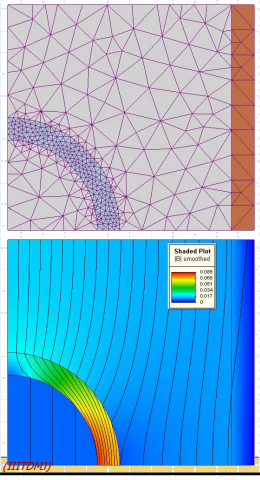
Applications

- Determine a triangulation then raise points to desired height



Applications

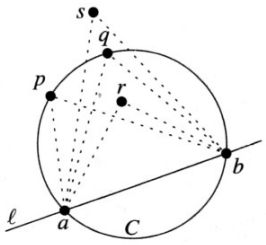
- Delaunay triangulations are used in creation of triangular meshes which are used analysis of stresses and strains in structures.



Thale's Theorem

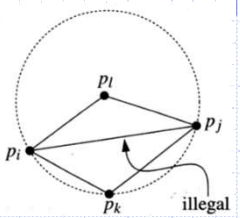
- *Thale's Theorem* can be used to test if an edge is legal
- Let *C* be a circle, l is a line intersecting *C* in points *a* and *b* and *p*, *q*, *r*, and *s* points lying on the same side of *l*. Suppose that *p* and *q* lie on *C*, that *r* lies inside *C*, and that *s* lies outside *C*. Then:

$$\angle arb > \angle apb = \angle aqb > \angle asb.$$



Testing for Illegal Edges

- If Pi, Pj, Pk, Pl form a convex quadrilateral and do not lie on a common circle, exactly one of PiPj and PkPl is an illegal edge
- The edge PiPj is illegal if Pl lies inside C
- Proved using Thales's Theorem. e.g., the angle Pi-Pk-Pj is smaller than the angle Pi-Pl-Pj



Computing Legal Triangulations

- Compute a triangulation of input points P
- Flip illegal edges of this triangulation until all edges are legal

Flip Algorithms

- If we have any triangulation of the points available, then illegal edges are found using Thale's Theorem and then they are flipped repeatedly until no triangle is non-Delaunay.

References:-

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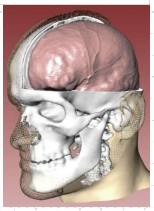
THANK YOU



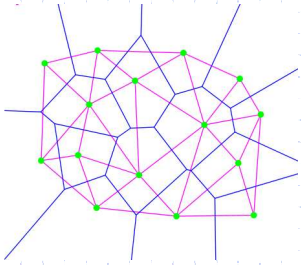
Dr. Prashant K. Jain
Professor (ME Discipline)
PDPM
INDIAN INSTITUTE OF INFORMATION TECHNOLOGY, DESIGN & MANUFACTURING JABALPUR
(An Institute of National Importance, established by MHRD, Govt. of India)
Ph. :+91-761-2794415
Cell :+91-94258-00310
Email:pkjain@iiitdmj.ac.in
pkjain2006 pkjain2006
Dumna Airport Road,
P.O. Khamaria
Jabalpur- 482 005, (M.P.), India
http://www.iiitdmj.ac.in

Incremental Algorithm

- Each point is inserted in Voronoi Diagram one by one.
- Step 1- Starting with the Voronoi diagram of $\{p_1 \ldots . p_i\}$ add point p_{i+1}
- Step 2- Explore all candidates to find the site p_j closest to p_{i+1} and compute its region by building its boundary starting from bisector $b_{i+1,j}$.
- Step 3- Update the diagram by pruning the initial edges.

Delaunay Triangulation

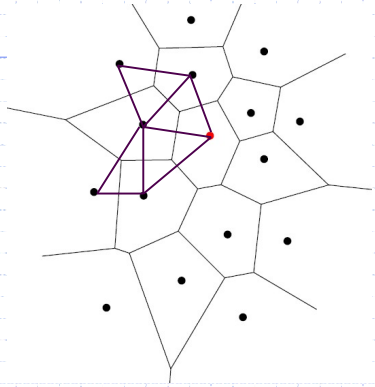
- Delaunay triangulation is a straight-line dual graph of the Voronoi Diagram of a point set.
- Maximize the minimum angle of all the angles of the triangles in the triangulation.
- Circumcircles of all the triangles are empty.



Voronoi Diagram
Delauny Triangulation

B. Delaunay: *Sur la sphère vide, Izvestia Akademii Nauk SSSR, Otdelenie Matematicheskikh i Estestvennykh Nauk, 7:793–800, 1934*

Delaunay Triangulation



Applications

- For modelling terrain or other objects given a set of sample points, the Delaunay triangulation gives a nice set of triangles to use as polygons in the model. In particular, the Delaunay triangulation avoids narrow triangles.

Applications

- Delaunay triangulations are used in creation of triangular meshes which are used for analysis of stresses and strains in structures.