Computer Aided Geometric Design

Projections

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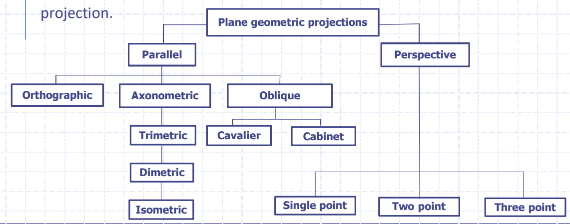
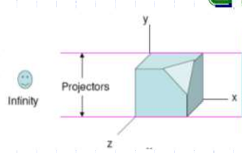
Geometric Projections

- Plane geometric projections of objects are formed by the intersection of lines called projectors with a plane called the projection plane.
- Projectors are lines from an arbitrary point called the center of projection, through each point in an object.
- If the center of projection is located at a finite point the result is a perspective projection.
- If it is at infinity, all the projectors are parallel and the result is parallel projection.

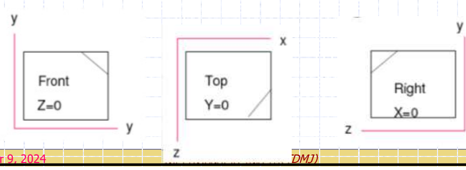


Orthographic Projections

- Commonly used for engineering drawings
- Shows true size and shape of object
- Projections onto one of the coordinate plane, x = 0, y = 0 or z = 0.

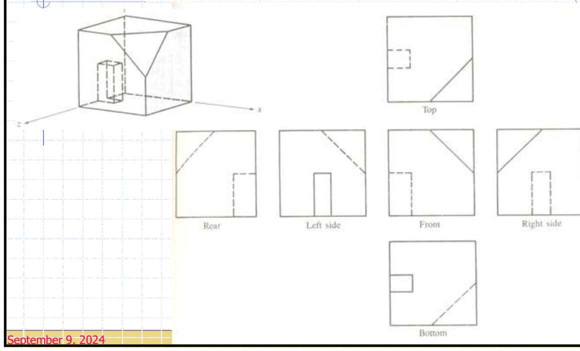


$$[P_z] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [P_y] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [P_x] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



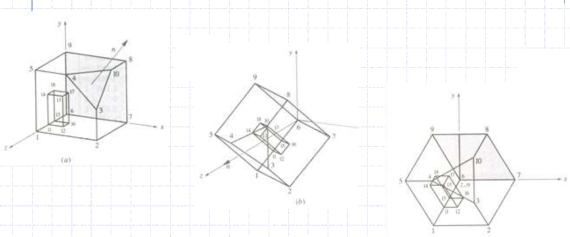
Multiview Orthographic Projections

Six view can be obtained by combinations of reflection, rotation and translation followed by projection onto z = 0 plane.



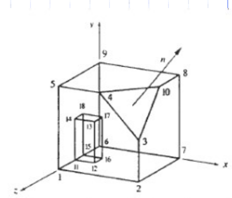
Auxiliary views

An auxiliary view is formed by rotating and translating the object so that the normal to the auxiliary plane is coincident with one of the coordinate axes.



- Develop an auxiliary view showing the true shape of the triangular corner for the object shown. The position vectors for the object are

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \\ 0 & 0 & 0.6 & 1 \\ 0.25 & 0 & 0.6 & 1 \\ 0.25 & 0.5 & 0.6 & 1 \\ 0 & 0.5 & 0.6 & 1 \\ 0 & 0 & 0.4 & 1 \\ 0.25 & 0 & 0.4 & 1 \\ 0.25 & 0.5 & 0.4 & 1 \\ 0 & 0.5 & 0.4 & 1 \end{bmatrix}$$



The vertex number shown in Figure, correspond to the rows in the position vector matrix [X].

The outward normal to the triangular face 3-4-10 is obtained using the position vectors 3, 4, 10. Specifically, taking the cross product of the vectors 3-10 and 3-4 prior to translation yields

n = ([10] – [3]) x ([4] – [3])

= [(1 - 1) (1 – 0.5) (0.5 - 1)] X [(0.5 – 1) (1 – 0.5) (1 – 1)]

= [0 1/2 -1/2] X [-1/2 1/2 0]

= [1/4 1/4 1/4]

Normalizing yields

Unit normal $n = [c_x \quad c_y \quad c_z] = [{}^1/_{\sqrt{3}} \quad {}^1/_{\sqrt{3}} \quad {}^1/_{\sqrt{3}}]$

Using previous Equations

$d = \sqrt{{c_y}^2 + {c_z}^2} \qquad \cos\alpha = \frac{c_z}{d} \qquad \sin\alpha = \frac{c_y}{d} \qquad \cos\beta = d, \quad \sin\beta = c_x$

Gives

$$d = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{2}{3}}$$

The normal is made coincident with the z-axis by rotation about the x-axis by an angle.

$$\alpha = \cos^{-1}\left(\frac{\frac{1}{\sqrt{3}}}{\frac{\sqrt{2}}{\sqrt{3}}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

followed by rotation about the y-axis by an angle

$$\beta = \cos^{-1}\left(\frac{2}{3}\right) = 35.26°$$

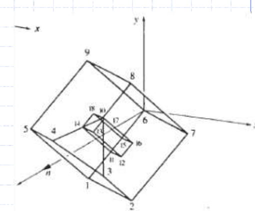
The concatenated transformation matrix is

$[T] = [R_x] [R_y]$

$$[T] = \begin{bmatrix} {}^2/_{\sqrt{6}} & 0 & {}^1/_{\sqrt{3}} & 0 \\ -{}^1/_{\sqrt{6}} & {}^1/_{\sqrt{2}} & {}^1/_{\sqrt{3}} & 0 \\ -{}^1/_{\sqrt{6}} & -{}^1/_{\sqrt{2}} & {}^1/_{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed position vectors are

$$[X'] = [X][T] = \begin{bmatrix} -0.408 & -0.707 & 0.577 & 1 \\ 0.408 & -0.707 & 1.155 & 1 \\ 0.204 & -0.354 & 1.443 & 1 \\ -0.408 & 0 & 1.443 & 1 \\ -0.816 & 0 & 1.155 & 1 \\ 0 & 0 & 0 & 1 \\ 0.816 & 0 & 0.577 & 1 \\ 0.408 & 0.707 & 10155 & 1 \\ -0.408 & 0.707 & 0.577 & 1 \\ 0.204 & 0.354 & 1.443 & 1 \\ -0.245 & -0.424 & 0.354 & 1 \\ -0.041 & -0.424 & 0.491 & 1 \\ -0.245 & -0.071 & 0.779 & 1 \\ -0.449 & -0.071 & 0.635 & 1 \\ -0.163 & -0.283 & 0.231 & 1 \\ 0.041 & -0.283 & 0.375 & 1 \\ -0.163 & 0.071 & 0.664 & 1 \\ -0.367 & 0.071 & 0.52 & 1 \end{bmatrix}$$

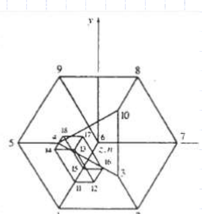


The auxiliary view is created by projecting this intermediate result onto the z = 0 plane, i.e.,

$$[P_z] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation matrices [T] and [Pz] are concentrated to yield

$$[T'] = [T][P_z] = \begin{bmatrix} {}^2/_{\sqrt{6}} & 0 & 0 & 0 \\ -{}^1/_{\sqrt{6}} & {}^1/_{\sqrt{2}} & 0 & 0 \\ -{}^1/_{\sqrt{3}} & -{}^1/_{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

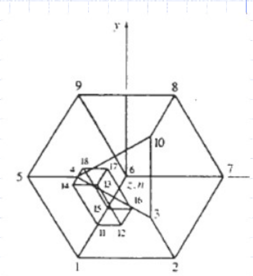


The auxiliary view is then created by

$$[X''] = [X][T']$$

[X''] is the same as [X'] except that the third column is all zeros, i.e., the effect of the projection is to neglect the z coordinate.

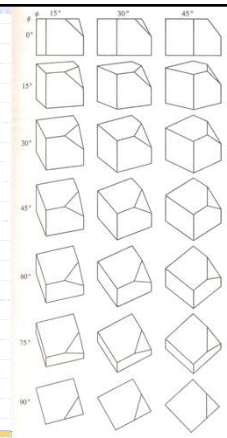
Notice the true shape of the triangle, which is equilateral.



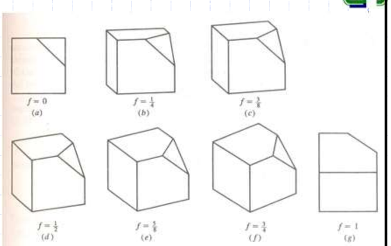
Axonometric projection

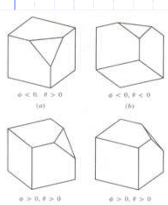
- An axonometric projection is constructed by manipulating the object, using rotations and translations, such that at least three adjacent faces are shown.
- Then projected from a center of projection at infinity onto one of the coordinate plane, normally the z=0 plane.
- Unless a face is parallel to the plane of projection, an axonometric projection does not show its true shape.
- Parallel lines equally foreshortened.
- Foreshortening factor is the ratio of the projected length of a line to its true length.
  - Trimetric
  - Dimetric
  - Isometric

Trimetric Projection

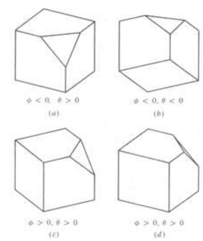


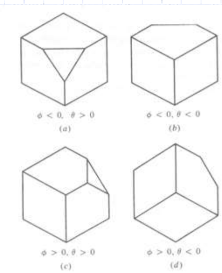
Dimetric Projection





Isometric Projection





Trimetric Projection

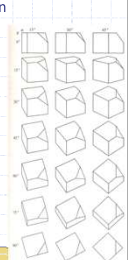
- A trimetric projection is formed at arbitrary rotations, in arbitrary order about any or all of the coordinate axes followed by parallel projection on to the z = 0 plane.
- Foreshortening ratio for each projected principal axis are all different in a general trimetric projection.
- Foreshortening ratios are obtained by applying concatenation

$$[U][T] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} [T] = \begin{bmatrix} x_x & y_x & 0 & 1 \\ x_y & y_y & 0 & 1 \\ x_z & y_z & 0 & 1 \end{bmatrix}$$

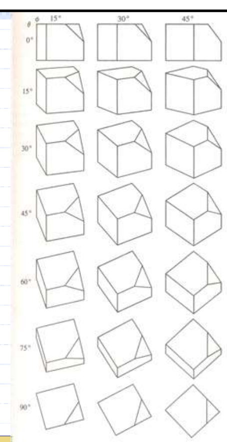
- [U] - unit vector along untransformed axes.
- [T] – concatenated trimetric projection matrix.

Then foreshortening factors

$$f_x = \sqrt{{x_x}^2 + {y_x}^2} \qquad f_y = \sqrt{{x_y}^2 + {y_y}^2} \qquad f_z = \sqrt{{x_z}^2 + {y_z}^2}$$



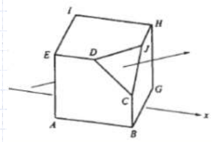
Trimetric Projection



Trimetric Projection

Findout a trimetric projection of cube with one corner removed by $\phi = 30^0$ rotation about the y-axis, followed by a $\theta = 45^0$ rotation about the x-axis, and then parallel projection onto the z = 0 plane. The position vectors for the cube with one corner removed are

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \end{bmatrix}$$



Trimetric Projection

The concatenated trimetric projection

[T] = [R_y] [R_x] [P_z]

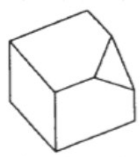
$$= \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi & -\sin\varphi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\varphi & -\cos\varphi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & \sqrt{2}/4 & 0 & 0 \\ 0 & \sqrt{2}/2 & 0 & 0 \\ 1/\sqrt{2} & -\sqrt{6}/4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Trimetric Projection

The transformed position vectors are

$$[X'] = [X][T] = \begin{bmatrix} 0.5 & -0.612 & 0 & 1 \\ 1.366 & -0.259 & 0 & 1 \\ 1.366 & 0.095 & 0 & 1 \\ 0.933 & 0.272 & 0 & 1 \\ 0.5 & 0.095 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.866 & 0.354 & 0 & 1 \\ 0.866 & 1.061 & 0 & 1 \\ 0 & 0.707 & 0 & 1 \\ 1.116 & 0.754 & 0 & 1 \end{bmatrix}$$

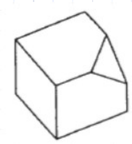


Trimetric Projection

The foreshortening ratios are

$$[U][T] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & \sqrt{2}/4 & 0 & 0 \\ 0 & \sqrt{2}/2 & 0 & 0 \\ 1/\sqrt{2} & -\sqrt{6}/4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & \sqrt{2}/4 & 0 & 1 \\ 0 & \sqrt{2}/2 & 0 & 1 \\ 1/\sqrt{2} & -\sqrt{6}/4 & 0 & 1 \end{bmatrix}$$



$$f_x = \sqrt{\left(\sqrt{3}/2\right)^2 + \left(\sqrt{2}/4\right)^2} = 0.935$$

$$f_y = \sqrt{2}/2 = 0.707$$

$$f_z = \sqrt{\left(1/2\right)^2 + \left(-\sqrt{6}/4\right)^2} = 0.791$$

Dimetric Projection

- Dimetric Projection is a special case of trimetric projection with two of the three foreshortening factors equal and third is arbitrary.
- Dimetric Projection is constructed by a rotation about y-axis through an angle ϕ followed by a rotation about x-axis through an angle θ and projection on z = 0 plane.
- Specific rotation angle can be obtained as:

Resulting Transformation is : T = R_y R_x P_z

$$[T] = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} \cos\varphi & \sin\varphi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\varphi & -\cos\varphi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Unit vectors on the x, y and z principal axes transform to

$$[U^*] = UT = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\varphi & -\cos\varphi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[U^*] = \begin{bmatrix} \cos\varphi & \sin\varphi\sin\theta & 0 & 1 \\ 0 & \cos\theta & 0 & 1 \\ \sin\varphi & -\cos\varphi\sin\theta & 0 & 1 \end{bmatrix}$$

$$f_x = \sqrt{{x_x}^2 + {y_x}^2} = \sqrt{\cos^2\varphi + \sin^2\varphi\sin^2\theta} \quad \text{---------------------(A)}$$

$$f_y = \sqrt{{x_y}^2 + {y_y}^2} = \sqrt{\cos^2\theta} \quad \text{---------------------(B)}$$

$$f_z = \sqrt{{x_z}^2 + {y_z}^2} = \sqrt{\sin^2\varphi + \cos^2\varphi\sin^2\theta} \quad \text{---------------------(C)}$$

Equating equation A and B

$$\cos^2\varphi + \sin^2\varphi\sin^2\theta = \cos^2\theta$$

$$1-\sin^2\varphi + \sin^2\varphi\sin^2\theta = 1-\sin^2\theta$$

$$\sin^2\varphi\,(1-\sin^2\theta) = \sin^2\theta$$

$$\sin^2\varphi = \frac{\sin^2\theta}{1-\sin^2\theta} \quad \text{-----------------------------(D)}$$

We know

$\cos^2\varphi = 1-\sin^2\varphi$

$\cos^2\theta = 1-\sin^2\theta$

Now using equation C and D

$$f_z{}^2 = \sin^2\varphi + \cos^2\varphi\sin^2\theta$$

$$f_z{}^2 = \frac{\sin^2\theta}{1-\sin^2\theta} + (1-\frac{\sin^2\theta}{1-\sin^2\theta})\sin^2\theta$$

$$f_z{}^2(1-\sin^2\theta) = \sin^2\theta + (1-2\sin^2\theta)\sin^2\theta$$

$$= \sin^2\theta + \sin^2\theta - 2\sin^4\theta$$

$$2\sin^4\theta - 2\sin^2\theta + f_z{}^2 - f_z{}^2\sin^2\theta = 0$$

$$2\sin^4\theta - (2+f_z{}^2)\sin^2\theta + f_z{}^2 = 0$$

$$2\sin^4\theta - (2+f_z{}^2)\sin^2\theta + f_z{}^2 = 0$$

Let $u = \sin^2\theta$ in this quadratic equation gives

$$\sin^2\theta = {f_z{}^2}/{2}, 1$$

$\mathrm{Sin}^2\theta = 1$ will give infinite result in Equation D, therefore discarded hence

$$\theta = \sin^{-1}\left(\pm {f_z}/{\sqrt{2}}\right)$$

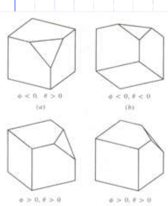
Putting this value in Eq. D will give

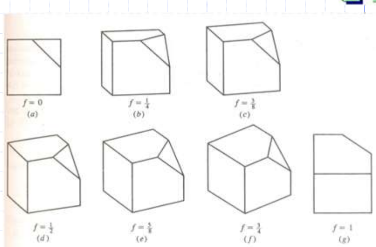
$$\varphi = \sin^{-1}\left(\pm {f_z}\Big/{\sqrt{2-f_z{}^2}}\right)$$

This shows that

$$0 \le f_z \le 1$$

Dimetric Projection



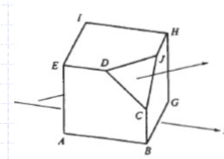


Each foreshortening factor f_z yields four possible dimetric projections.

Dimetric Projection

- For the cube with the corner cut off, determine the dimetric projection for a foreshortening factor along the z-axis of 1/2. take parallel projection onto the z = 0 plane.
- The position vectors for the cube with one corner removed are

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \end{bmatrix}$$



Dimetric Projection

Angle of rotation about x-axis

$$\theta = \sin^{-1}\left(\pm {f_z}/{\sqrt{2}}\right)$$

$$= \sin^{-1}\left(\pm {1}/{2\sqrt{2}}\right)$$

$$= \sin^{-1}(\pm 0.35355) \quad = \pm 20.705^0$$

And rotation angle about y-axis

$$\varphi = \sin^{-1}\left(\pm {f_z}\Big/{\sqrt{2-f_z{}^2}}\right)$$

$$= \sin^{-1}\left[\pm\frac{(1/2)}{\sqrt{7/4}}\right]$$

$$= \sin^{-1}(\pm 0.378) \quad = \pm 22.208^0$$

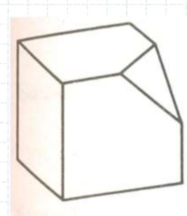
Dimetric Projection

Choosing $\phi = +22.208^0$ and $\theta = +20.705^0$, The dimetric projection matrix

$$T = \begin{bmatrix} \cos\varphi & \sin\varphi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\varphi & -\cos\varphi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.926 & 0.134 & 0 & 0 \\ 0 & 0.935 & 0 & 0 \\ 0.378 & -0.327 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed position vectors are

$$[X'] = [X][T] = \begin{bmatrix} 0.378 & -0.327 & 0 & 1 \\ 1.304 & -0.194 & 0 & 1 \\ 1.304 & 0.274 & 0 & 1 \\ 0.841 & 0.675 & 0 & 1 \\ 0.378 & 0.608 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.926 & 0.134 & 0 & 1 \\ 0.926 & 1.069 & 0 & 1 \\ 0 & 0.935 & 0 & 1 \\ 1.115 & 0.905 & 0 & 1 \end{bmatrix}$$



$f = \frac{1}{2}$

Isometric Projection

- Isometric Projection is a special case of trimetric projection with all three foreshortening factors equal.
- Isometric Projection is constructed by a rotation about y-axis through an angle ϕ followed by a rotation about x-axis through an angle θ and projection on z = 0 plane.
- Specific rotation angle can be obtained as:

Resulting Transformation is : T = R_y R_x P_z

$$[T] = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} \cos\varphi & \sin\varphi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\varphi & -\cos\varphi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Isometric Projection

Unit vectors on the x, y and z principal axes transform to

$$[U^*] = UT = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\varphi & -\cos\varphi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[U^*] = \begin{bmatrix} \cos\varphi & \sin\varphi\sin\theta & 0 & 1 \\ 0 & \cos\theta & 0 & 1 \\ \sin\varphi & -\cos\varphi\sin\theta & 0 & 1 \end{bmatrix}$$

$$f_x = \sqrt{{x_x}^2 + {y_x}^2} = \sqrt{\cos^2\varphi + \sin^2\varphi\sin^2\theta} \quad \text{--------------------(A)}$$

$$f_y = \sqrt{{x_y}^2 + {y_y}^2} = \sqrt{\cos^2\theta} \quad \text{--------------------(B)}$$

$$f_z = \sqrt{{x_z}^2 + {y_z}^2} = \sqrt{\sin^2\varphi + \cos^2\varphi\sin^2\theta} \quad \text{--------------------(C)}$$

Isometric Projection

Equating equation A and B

$$\cos^2\varphi + \sin^2\varphi\sin^2\theta = \cos^2\theta$$

$$1 - \sin^2\varphi + \sin^2\varphi\sin^2\theta = 1 - \sin^2\theta$$

$$\sin^2\varphi\,(1 - \sin^2\theta) = \sin^2\theta$$

We know

$\cos^2\varphi = 1 - \sin^2\varphi$

$\cos^2\theta = 1 - \sin^2\theta$

$$\sin^2\varphi = \frac{\sin^2\theta}{1 - \sin^2\theta} \quad \text{-----------------------------(D)}$$

Now equate equation B and C:

We know

$\cos^2\varphi = 1 - \sin^2\varphi$

$\cos^2\theta = 1 - \sin^2\theta$

$$\cos^2\theta = \sin^2\varphi + \cos^2\varphi\sin^2\theta$$

$$1 - \sin^2\theta = \sin^2\varphi + (1 - \sin^2\varphi)\sin^2\theta$$

$$1 - \sin^2\theta = \sin^2\varphi + \sin^2\theta - \sin^2\varphi\sin^2\theta$$

$$1 - 2\sin^2\theta = \sin^2\varphi\,(1 - \sin^2\theta)$$

$$\sin^2\varphi = \frac{1 - 2\sin^2\theta}{1 - \sin^2\theta} \quad \text{-----------------------------(E)}$$

Isometric Projection

Now Equating equation D and E

$$\frac{\sin^2\theta}{1 - \sin^2\theta} = \frac{1 - 2\sin^2\theta}{1 - \sin^2\theta}$$

Gives

$$\sin^2\theta = \frac{1}{3}$$

$$\sin\theta = \pm\frac{1}{\sqrt{3}}$$

$$\theta = \pm 35.26^0$$

Using equation D

$$\sin^2\varphi = \frac{1/3}{1 - 1/3} = 1/2 \qquad \varphi = \pm 45^0$$

Then using equation B

$$f_z = \sqrt{\cos^2\theta} = \sqrt{2/3} = 0.8165$$

Isometric Projection

The angle: projected x-axis makes with the horizontal is important for manual construction.

Transforming the unit vector along the x-axis

$$[U_x^*] = [1 \quad 0 \quad 0 \quad 1] \begin{bmatrix} \cos\varphi & \sin\varphi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\varphi & -\cos\varphi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[U_x^*] = [\cos\varphi \quad \sin\varphi\sin\theta \quad 0 \quad 1]$$

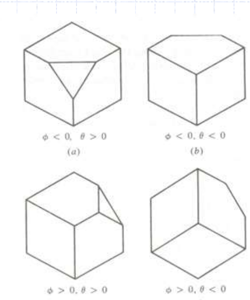
The angle between projected x-axis and the horizontal is then

$$\tan\alpha = \frac{y_x^*}{x_x^*} = \frac{\sin\varphi\sin\theta}{\cos\varphi} = \pm\sin\theta$$

Since sin ∅ = cos ∅ for ∅ = 45°, then α is

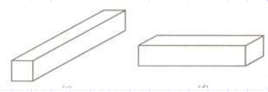
$$\alpha = \tan^{-1}(\pm\sin 3\,5.26) = \pm 30^0$$

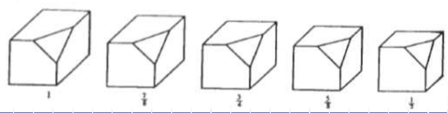
Isometric Projection



Oblique Projection

- Oblique projection is formed by parallel projectors from a center of projection at infinity that intersect the plane of projection at an oblique angle.
- Illustrate the general three dimensional shape of the object.
- Cavalier and Cabinet

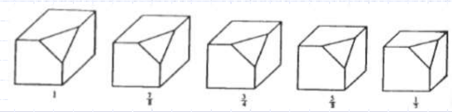




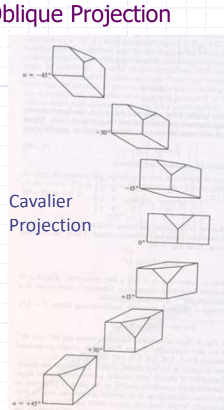
Oblique Projection

- Cavalier
  - Foreshortening factor is equal for all three axes and one.
  - Angle between oblique projectors and plane of projection is 45^0.
  - Resulting figure appear too thick.
- Cabinet
  - Foreshortening factor for edges perpendicular to plane of projection is one half.
  - Angle between oblique projectors and plane of projection is $\cot^{-1}(1/2) = 63.43^0$.

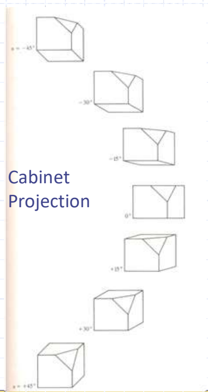
$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f\cos\alpha & -f\sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Oblique Projection



Cavalier Projection



Cabinet Projection

Perspective Transformation

- Perspective transformation is transformation from one three space to another three space.
- Parallel lines converge, object size reduced with increasing distance from center to projection.
- Aid the depth perception, but shape is not preserved.
- A single point PT is given by

$$[x \quad y \quad z \quad 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x \quad y \quad z \quad rz+1]$$

Here h = rz+1 $\neq$ 1

$$[x' \quad y' \quad z' \quad 1] = \left[\frac{x}{rz+1} \quad \frac{y}{rz+1} \quad \frac{z}{rz+1} \quad 1\right]$$

Followed by projection onto z = 0 plane

$$[x' \quad y' \quad z' \quad 1] = \left[\frac{x}{rz+1} \quad \frac{y}{rz+1} \quad 0 \quad 1\right]$$

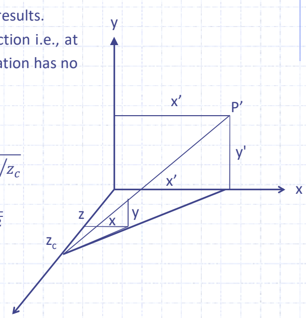
Perspective Transformation

- let $r = -1/z_c$
- If z_c approaches infinity, r approaches zero and an axonometric projection results.
- If a point on the plane of projection i.e., at z = 0 the perspective transformation has no effect.
- Origin is unaffected.

$$\frac{x'}{z_c} = \frac{x}{z_c - z} \qquad x' = \frac{x}{1 - z/z_c}$$

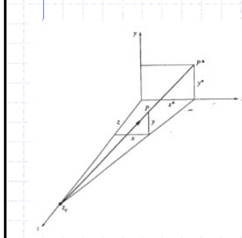
$$\frac{y'}{\sqrt{x'^2 + z_c^2}} = \frac{y}{\sqrt{x^2 + (z_c - z)^2}}$$

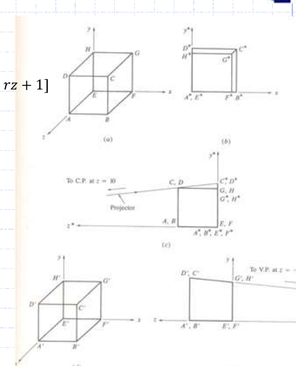
$$y' = \frac{y}{1 - z/z_c} \qquad z' = 0$$



Single point perspective projection

$$[x \quad y \quad z \quad 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x \quad y \quad z \quad rz+1]$$

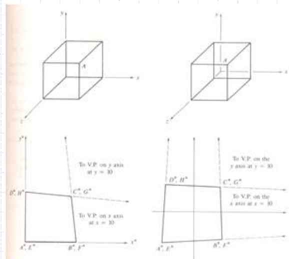




Two point perspective projection

$$[x \quad y \quad z \quad 1]\begin{bmatrix}1 & 0 & 0 & p\\ 0 & 1 & 0 & q\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} = [x \quad y \quad z \quad (px+qy+1)]$$

$$[x' \quad y' \quad z' \quad 1] = \left[\frac{x}{px+qy+1} \quad \frac{y}{px+qy+1} \quad \frac{z}{px+qy+1} \quad 1\right]$$



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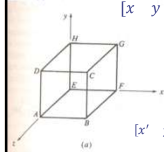
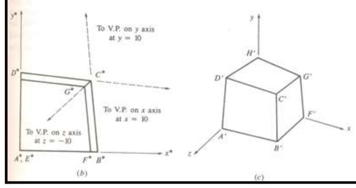
Three point perspective projection

$$[x \quad y \quad z \quad 1]\begin{bmatrix}1 & 0 & 0 & p\\ 0 & 1 & 0 & q\\ 0 & 0 & 1 & r\\ 0 & 0 & 0 & 1\end{bmatrix} = [x \quad y \quad z \quad (px+qy+rz+1)]$$



$$[x' \quad y' \quad z' \quad 1] = \left[\frac{x}{px+qy+rz+1} \quad \frac{y}{px+qy+rz+1} \quad \frac{z}{px+qy+rz+1} \quad 1\right]$$



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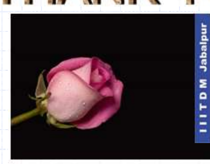
THANK YOU



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