Computer Aided Geometric Design

Curves

Representation of Parametric curves

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INDIAN INSTITUTE OF INFORMATION TECHNOLOGY, DESIGN AND MANUFACTURING JABALPUR
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Parametric curves

Curves can be classified in number of ways:

- Plane curves & Space curves
  - Example: Circle & Helix
- Curves of known forms & Free-form curves
  - Example: Circle vs. Bezier Curve
- Interpolation curves and Approximation curves
  - Example: Hermite Curve vs. Bezier Curve

Parametric representation of line

Line in xy-plane

x = x1 + (x2 - x1) u

y = y1 + (y2 - y1) u

$0 \le u \le 1.0$

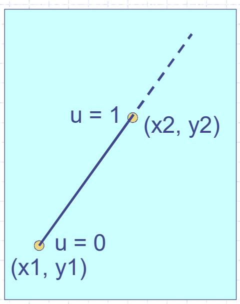
Line in space

x = x1 + (x2 - x1) u

y = y1 + (y2 - y1) u

z = z1 + (z2 - z1) u

$0 \le u \le 1.0$



Parametric representation of circle

Circle in xy-plane

x = x1 + r cos (θ)

y = y1 + r sin (θ)

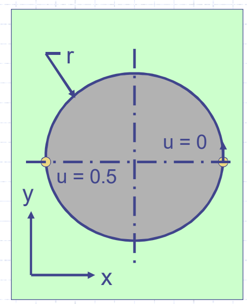
$0 \le \theta \le 2\pi$

Circle in xy-plane

x = x1 + r cos (2π u)

y = y1 + r sin (2π u)

$0 \le u \le 1$



Parametric representation of circle

Circular Arc

x = x1 + r cos (θ)

y = y1 + r sin (θ)

$0 \le \theta \le \pi/2$

Circular Arc

x = x1 + r cos ($\pi/2$ u)

y = y1 + r sin ($\pi/2$ u)

$0 \le u \le 1$

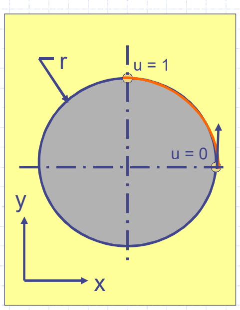
Circular Helix

x = x1 + r cos (2π u)

y = y1 + r sin (2π u)

z = h u

$0 \le u \le 1$



Conic sections

Represents a conic section curve

Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$
Parabola	$y^2-4ax=0$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$

Parametric representation of conics

Ellipse

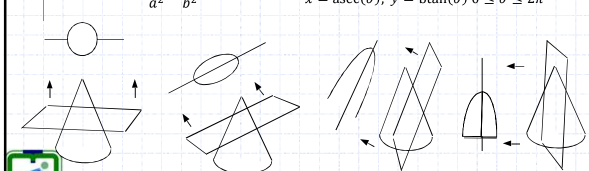
$x = a\cos(\theta);\ y = b\sin(\theta)\ 0 \le \theta \le 2\pi$

Parabola

$x = au^2;\ y = 2au\ \ 0 \le u \le 1$

Hyperbola

$x = a\sec(\theta);\ y = b\tan(\theta)\ 0 \le \theta \le 2\pi$



Circle Ellipse Parabola Hyperbola

Implicit/Explicit Vs. Parametric

Implicit Representation	Explicit Representation	Parametric Representation
$x^2 + y^2 - r^2 = 0$	$y = (r^2 - x^2)^{1/2}$	$x = r \cos(t)$ $y = r \sin(t)$
$ax + by + cz + d = 0$	$z = px + qy + r$	$x = a + bu + cw$ $y = d + eu + fw$ $z = g + hu + iw$

f(x,y)=0 y=f(x)

Parametric Representation

Parametric equations completely separate the role of dependent and independent variables

What do these equations represent?

$x = r \cos(t)$
$y = r \sin(t)$
$z = h$

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$x = r \cos(t)$
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$r = 5;\ t = \pi;\ h = 20;$

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Parametric Representation

Parametric equations completely separate the role of dependent and independent variables

What do these equations represent?

$x = r \cos(t)$
$y = r \sin(t)$
$z = h$

$0 \le r \le 5;\ 0 \le t \le 2\pi;\ 0 \le h \le 20;$

Parametric Representation

- Offers more degree of freedom for controlling the shape of curves and surfaces

Explicit form

$y = px^3 + qx^2 + rx + s$

Parametric form

$x = au^3 + bu^2 + cu + d$

$y = eu^3 + fu^2 + gu + h$

Parametric Representation

- Transformations are easier to apply

Circle with center (0,0) and radius 7 units	Circle with center (4,3) and radius 7 units
$x = 7\cos(t)$ $y = 7\sin(t)$	$x = 4 + 7\cos(t)$ $y = 3 + 7\sin(t)$
$x^2 + y^2 - 49 = 0$	$x^2 + y^2 - 8x - 6y - 24 = 0$

Parametric Representation

- Advantage in representation of curve and surface segments

Circle	Circular arc
$x = r\cos(t)$ $y = r\sin(t)$	$x = r\cos(t)$ $y = r\sin(t)$
$r = 5;\ 0 \le t \le 2\pi;$	$r = 5;\ 0 \le t \le \pi;$

Parametric Representation

- Advantage in handling infinite slope

Implicit/Explicit $dy/dx = \infty$

Parametric $= \dfrac{dy/du}{dx/du} = \infty$

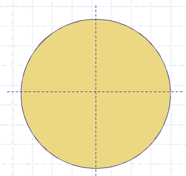
implies $dx/du = 0$

Parametric Representation

- Advantage in calculation of points for display and tool path

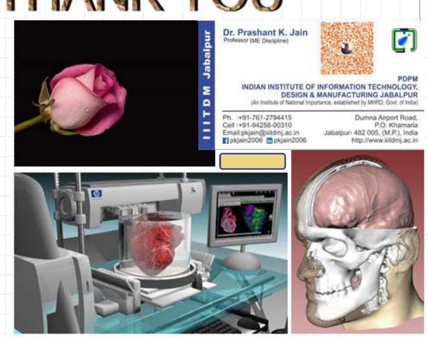
$x^2 + y^2 - 49 = 0$
(Implicit)

$x = 7\cos(t)$
$y = 7\sin(t)$
(Parametric)



How to approximate circle with lines

THANK YOU



Computer Aided Geometric Design

Curves

Parametric cubic curve

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Parametric Representation of Free-form Curves

Interpolation Curves

- Parametric Cubic Curve

Approximation Curves

- Bezier Curve
- B-Spline Curve
- NURBS Curve

Parametric Cubic Curve

- It is also known as Hermite Curve
- It is an Interpolation Curve
- It has three different forms
  - Algebraic Form (12 algebraic coefficients)
  - Geometric Form (End points & tangent vectors)
  - Four - Point Form (Four points)

Algebraic and Geometric Form

- Parametric cubic (PC) curve segment

$$\left.\begin{aligned} x(u) &= a_{3x}u^3 + a_{2x}u^2 + a_{1x}u + a_{0x} \quad &(1)\\ y(u) &= a_{3y}u^3 + a_{2y}u^2 + a_{1y}u + a_{0y} \quad &(2)\\ z(u) &= a_{3z}u^3 + a_{2z}u^2 + a_{1z}u + a_{0z} \quad &(3)\end{aligned}\right\} \text{12 unknowns}$$

- 12 constant coefficients called algebraic coefficients
- Two same curves of same shape have different algebraic coefficients if they occupy different position in space.
- In vector notation

$$P(u) = a_3u^3 + a_2u^2 + a_1u + a_0 \qquad (4)$$

P(u) is position vector

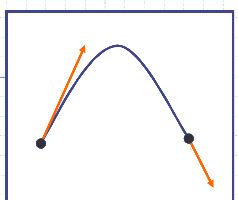
Algebraic and **Geometric** Form

Algebraic coefficients are not convenient for controlling shape of curve and in intuitive sense of curve

Geometric Form

starting point
(x_0, y_0, z_0)
end point
(x_1, y_1, z_1)

starting tangent vector
($x`_0$, $y`_0$, $z`_0$)
end tangent vector
($x`_1$, $y`_1$, $z`_1$)



$$x`_0 = \left.\frac{dx}{du}\right|_{u=0}$$

Algebraic and **Geometric Form**

- For geometric form, using two end points P(0) and P(1) and corresponding tangent vectors P'(0) and P'(1) we obtain four equations from Eq. (4).

(Substitute u = 0 & u = 1)

$$\left.\begin{aligned} p(0) &= a_0\\ p(1) &= a_0 + a_1 + a_2 + a_3\\ p'(0) &= a_1\\ p(1) &= a_0 + a_1 + a_2 + a_3\end{aligned}\right\} \qquad (5)$$

- By solving

$$a_0 = p(0)$$
$$a_1 = p'(0)$$
$$a_2 = -3p(0) + 3p(1) - 2p'(0) - p'(1)$$
$$a_3 = 2p(0) - 2p(1) + p'(0) + p'(1)$$

Then eq. (4) can be written as

$$p(u) = (2u^3 - 3u^2 + 1)P(0) + (-2u^3 + 3u^2)P(1) + (u^3 - 2u^2 + u)P'(0) + (u^3 - u^2)P'(1)$$

Further equation can be simplified as

$$a_2 = -3p(0) + 3p(1) - 2p'(0) - p'(1) \quad \text{———} \quad (5)$$

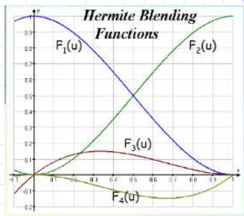
where

$$F_1 = 2u^3 - 3u^2 + 1$$

$$F_2 = -2u^3 + 3u^2$$

$$F_3 = u^3 - 2u^2 + u$$

$$F_4 = u^3 - u^2$$



P_0, P_1, P'_0 and P'_1 are called geometric coefficients.

F terms are called blending functions also called hermite basis functions.

Algebraic equation can be written In matrix form

$$P(u) = [u^3 \quad u^2 \quad u \quad 1][a_3 \quad a_2 \quad a_1 \quad a_0]^T$$

$$P(u) = [U][A] \quad \text{———} \quad (6)$$

Similarly for geometric form

$$P(u) = [F_1 \quad F_2 \quad F_3 \quad F_4][P_0 \quad P_1 \quad P'_0 \quad P'_1]^T$$

$$a_1 = p'(0)$$

Then

$$F = [(2u^3 - 3u^2 + 1) \quad (-2u^3 + 3u^2) \quad (u^3 - 2u^2 + u) \quad (u^3 - u^2)]$$

Leads to

$$F = [u^3 \quad u^2 \quad u \quad 1]\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= UM_F$$

$$F = UM_F$$

$$P(u) = UM_F B \quad \text{———} \quad (7)$$

$$P(u) = [U][A] \quad \text{———} \quad (6)$$

$$P(u) = UM_F B \quad \text{———} \quad (7)$$

From equation 6 and 7

$$A = M_F B$$

and

$$B = M_F^{-1} A$$

where

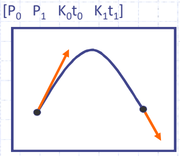
$$M_F^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

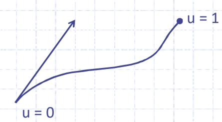
This gives conversion between algebraic and geometric forms.

Tangent Vector

- The magnitude or length of the tangent vector depends on parameterization and affects the interior shape of the curve.
- Specifying coordinates and slopes at end points of a cubic hermite curve accounts for only 10 of the 12 dof
- Six are from x_0 y_0 z_0 and x_1 y_1 z_1. Four more from direction cosines, two from each end. This means there are two more degrees of freedom to control the shape of a curve.
- We can express the matrix of geometric coefficients as

B = [P_0 P_1 K_0t_0 K_1t_1]





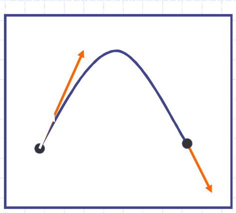
PARAMETRIC CUBIC CURVE

Tangent Vectors

($x`_0$, $y`_0$, $z`_0$)
($x`_1$, $y`_1$, $z`_1$)

can be written as

(k_0l_0, k_0m_0, k_0n_0)
(k_1l_1, k_1m_1, k_1n_1)



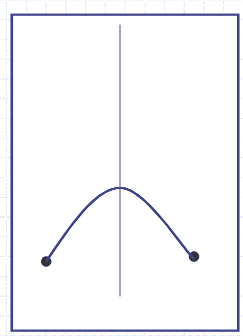
(l_0, m_0, n_0) & (l_1, m_1, n_1) are direction cosines of tangent vector at start & end points

PARAMETRIC CUBIC CURVE

Tangent Vectors

(k_0l_0, k_0m_0, k_0n_0)
(k_1l_1, k_1m_1, k_1n_1)

Effect of Increasing k_0 & k_1 on Curve Shape

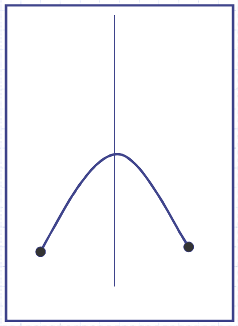


PARAMETRIC CUBIC CURVE

Tangent Vectors

$(k_0 l_0,\ k_0 m_0,\ k_0 n_0)$
$(k_1 l_1,\ k_1 m_1,\ k_1 n_1)$

Effect of Increasing k_0 & k_1 on Curve Shape

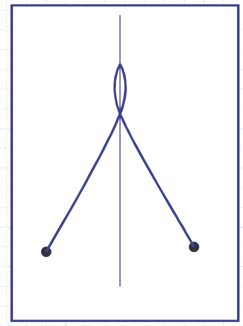


PARAMETRIC CUBIC CURVE

Tangent Vectors

$(k_0 l_0,\ k_0 m_0,\ k_0 n_0)$
$(k_1 l_1,\ k_1 m_1,\ k_1 n_1)$

Effect of Increasing k_0 & k_1 on Curve Shape

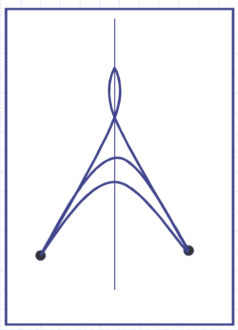


PARAMETRIC CUBIC CURVE

Tangent Vectors

$(k_0 l_0,\ k_0 m_0,\ k_0 n_0)$
$(k_1 l_1,\ k_1 m_1,\ k_1 n_1)$

Effect of Increasing k_0 & k_1 on Curve Shape

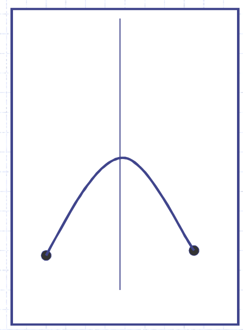


PARAMETRIC CUBIC CURVE

Tangent Vectors

$(k_0 l_0,\ k_0 m_0,\ k_0 n_0)$
$(k_1 l_1,\ k_1 m_1,\ k_1 n_1)$

Effect of Increasing k_1 relative to k_0 on Curve Shape



PARAMETRIC CUBIC CURVE

Tangent Vectors

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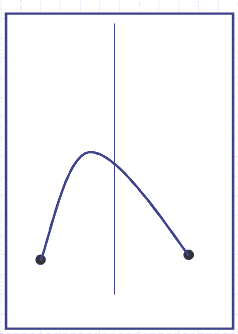
$(k_1 l_1,\ k_1 m_1,\ k_1 n_1)$

Effect of Increasing k_1 relative to k_0 on Curve Shape

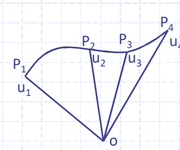


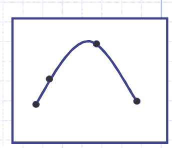
Four point form

Four point in space

$[P_1\ \ P_2\ \ P_3\ \ P_4]$

$u_1 < u_2 < u_3 < u_4$





Now find a matrix of constants which produces B matrix when pre multiplies with coordinate matrix of four points thus

$$[B] = K[P_1 \quad P_2 \quad P_3 \quad P_4]^T$$

$$[P_0 \quad P_1 \quad P_0' \quad P_1'] = K[P_1 \quad P_2 \quad P_3 \quad P_4]^T$$

$$[P_1 \quad P_2 \quad P_3 \quad P_4]^T = [U_1 \quad U_2 \quad U_3 \quad U_4]^T MB$$ From (7)

where

$$U_1 = [u_1{}^3 \quad u_1{}^2 \quad u \quad 1]$$

$$[U_1 \quad U_2 \quad U_3 \quad U_4]^T = \begin{bmatrix} u_1^3 & u_1^2 & u_1 & 1 \\ u_2^3 & u_2^2 & u_2 & 1 \\ u_3^3 & u_3^2 & u_3 & 1 \\ u_4^3 & u_4^2 & u_4 & 1 \end{bmatrix}$$

For

$$B = M^{-1}[U_1 \quad U_2 \quad U_3 \quad U_4]^{T-1}[P_1 \quad P_2 \quad P_3 \quad P_4]^T$$

Hence

$$K = M^{-1}[U_1 \quad U_2 \quad U_3 \quad U_4]^{T-1}$$

Now

$$B = K[P_1 \quad P_2 \quad P_3 \quad P_4]^T$$

Conversely

$$[P_1 \quad P_2 \quad P_3 \quad P_4]^T = K^{-1}B$$

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If we chose equally distributed u values that $u_1=0$, $u_2=1/3$, $u_3=2/3$, $u_4=1$ then K and K^{-1} will be

$$K = M^{-1}U^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/27 & 1/9 & 1/3 & 1 \\ 8/27 & 4/9 & 2/3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -11/2 & 9 & -9/2 & 1 \\ -1 & 9/2 & -9 & 11/2 \end{bmatrix}$$

$$K^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 20/27 & 7/27 & 4/27 & -2/27 \\ 7/27 & 20/27 & 2/27 & -4/27 \\ 0 & 1 & 0 & 0 \end{bmatrix} = UM$$

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We can also have

let $[P_1 \quad P_2 \quad P_3 \quad P_4] = [U_1 \quad U_2 \quad U_3 \quad U_4]^T MB$

Since p = UMB and B=KP

hence p=UMKP

Let MK = N

Hence p = UNP

$$N = \begin{bmatrix} -9/2 & 27/2 & -27/2 & 9/2 \\ 9 & -45/2 & 18 & -9/2 \\ -11/2 & 9 & -9/2 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Algebraic coefficients are found from A = NP

$p = (-4.5u^3 + 9u^2 - 5.5u + 1) P_1$

$+ (13.5u^3 - 22.5u^2 + 9u) P_2$

$+(-13.5u^3 + 18u^2 - 4.5u) P_3$

$+(4.5u^3 - 4.5u^2 + u) P_4$

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Question 6

- A cubic Hermite curve defined in four point form by A1[4 5 4], A2[6 7 4], A3[8 7 4] and A4[7 5 4] at u1=0, u2=1/4, u3=3/4, u4=1 find its geometric coefficients and a point at u=1/2.

$$[P_1 \quad P_2 \quad P_3 \quad P_4]^T = \begin{bmatrix} 4 & 5 & 4 \\ 6 & 7 & 4 \\ 8 & 7 & 4 \\ 7 & 5 & 4 \end{bmatrix}$$

$$u_1 = 0, u_2 = \frac{1}{4}, u_2 = \frac{3}{4}, u_2 = 1$$

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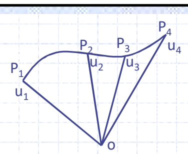
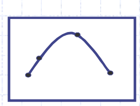
Four point form





Four point in space

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where

$$[P_1 \quad P_2 \quad P_3 \quad P_4]^T = [U_1 \quad U_2 \quad U_3 \quad U_4]^T MB$$ ——From (7)

$$U_1 = [u_1{}^3 \; u_1{}^2 \; u \; 1]$$

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$$[U_1 \quad U_2 \quad U_3 \quad U_4]^T = \begin{bmatrix} u_1{}^3 & u_1{}^2 & u_1 & 1 \\ u_2{}^3 & u_2{}^2 & u_2 & 1 \\ u_3{}^3 & u_3{}^2 & u_3 & 1 \\ u_4{}^3 & u_4{}^2 & u_4 & 1 \end{bmatrix}$$

For $[B] = M^{-1}[U_1 \quad U_2 \quad U_3 \quad U_4]^{T-1}[P_1 \quad P_2 \quad P_3 \quad P_4]^T$

Hence $K = M^{-1}[U_1 \quad U_2 \quad U_3 \quad U_4]^{T-1}$

Now $B = K[P_1 \quad P_2 \quad P_3 \quad P_4]^T$

Conversely $[P_1 \quad P_2 \quad P_3 \quad P_4]^T = M^{-1}B$

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If we chose equally distributed u values that $u_1=0$, $u_2=1/4$, $u_3=3/4$, $u_4=1$ then K and K^{-1} will be

$$K = M^{-1}U^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{64} & \frac{1}{16} & \frac{1}{4} & 1 \\ \frac{27}{64} & \frac{9}{16} & \frac{3}{4} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -6.33 & 8 & -2.66 & 1 \\ -1 & 2.66 & -8 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -6.33 & 8 & -2.66 & 1 \\ -1 & 2.66 & -8 & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 & 4 \\ 6 & 7 & 4 \\ 8 & 7 & 4 \\ 7 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 4 \\ 7 & 5 & 4 \\ 8.33 & 10.66 & 0 \\ -7.66 & -10.67 & 0 \end{bmatrix}$$

point at u=1/2

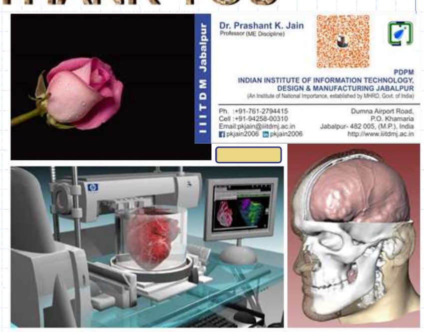
$$P = [U_1 \quad U_2 \quad U_3 \quad U_4]^T MB$$

$$P = [{u_0}^3 \ \ {u_0}^2 \ \ u_0 \ \ 1]MB$$

$$P = [0.5^3 \ \ 0.5^2 \ 0.5 \ 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 5 & 4 \\ 7 & 5 & 4 \\ 8.33 & 10.66 & 0 \\ -7.66 & -10.67 & 0 \end{bmatrix}$$

$$P_{0.5} = [7.5 \quad 7.667 \quad 4]$$

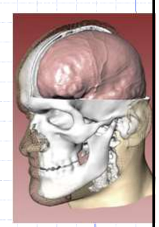
THANK YOU



Computer Aided Geometric Design

Curves

Re-parameterization
Truncating, extending, subdividing of curves
Composite Curves, Continuity conditions



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Re-Parameterization

- Change in parametric interval so that neither the shape nor position of the curve change

  v = f(u)

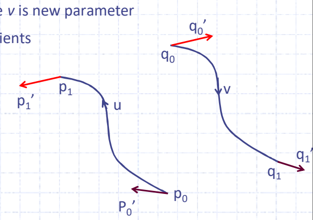
  for changing the direction of curve

  v = -u where *v* is new parameter

  B matrix of geometric coefficients

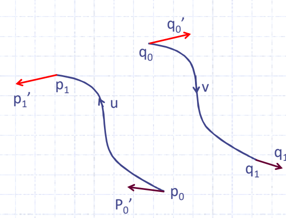
  $B_1 = [p_0 \ \ p_1 \ \ p_0' \ \ p_1']^T$

  $B_2 = [q_0 \ \ q_1 \ \ q_0' \ \ q_1']^T$



Re-Parameterization

$$\begin{bmatrix} p_0 \\ p_1 \\ p_0' \\ p_1' \end{bmatrix} \Rightarrow \begin{bmatrix} q_0 \\ q_1 \\ q_0' \\ q_1' \end{bmatrix}$$



If

$q_0 = p_1$

$q_1 = p_0$

$q_0' = -p_1'$

$q_1' = -p_0'$

Then two curves are identical except that they have opposite directions of parameterization

Then $B_2 = [P_1 \ \ P_0 \ \ -P_1' \ \ -P_0']$

Re-Parameterization

- For a more general case curve is initially parameterized from u_i to u_j and want to change parametric variable ranges from v_i to v_j.
- Let $B_1 = [p_i \;\; p_j \;\; p_i' \;\; p_j']^T$ and $B_2 = [q_i \;\; q_j \;\; q_i' \;\; q_j']^T$
- The end points are invariant or insensitive to any change of parameterization, so $q_i = p_i$ and $q_j = p_j$ to maintain constant position.
- Tangent vectors are sensitive to the functional relationship between u and v, i.e. v = f(u), a linear relationship is required to preserve cubic form.

Thus $v = au + b$ then $dv = a\,du$

also $v_i = au_i + b$ and $v_j = au_j + b$ from this a and b can be found and relationship between tangent vectors

$$q' = \frac{u_j - u_i}{v_j - v_i} p'$$

Re-Parameterization

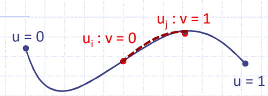
Now complete relationship between two sets of geometric coefficients

$$q_i = p_i$$
$$q_j = p_j$$
$$q_i' = \frac{u_j - u_i}{v_j - v_i} p_i'$$
$$q_j' = \frac{u_j - u_i}{v_j - v_i} p_j'$$

- This also shows that tangent vector magnitudes must change to accommodate a change in the range of the parametric variable.
- Magnitudes are simply scaled by the ratio of the ranges of parametric variable.
- This preserves the direction of tangent vectors and shape of the curve.

Truncating, Extending and Subdividing

- Find new B matrix for a truncated or extended pc curve



- curve truncated at u_i and u_j from segment u_0 to u_i and from u_j to u_1. we can represent remaining curve segment as a full pc curve from v_0 to v_1.
- Compute p_i and p_j using p = UMB and p_i' and p_j' using p' = UM'B.
- Ratio of parametric interval length $(u_j - u_i)/(v_j - v_i)$ reduces to $u_j - u_i$, since $v_j - v_i = 1$. Then

$$q_0 = p_i \qquad q_1 = p_j$$
$$q_0' = (u_j - u_i)p_i' \qquad q_1' = (u_j - u_i)p_j'$$

- To subdivide a pc curve into *n* successive segments of arbitrary length and generate n new pc curves, then geometric coefficient or B matrix of i^{th} segment are

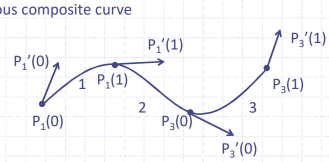
$$B_i = [p_{i-1} \quad p_i \quad (u_i - u_{i-1})p'_{i-1} \quad (u_i - u_{i-1})p'_i]$$

- If a curve is divided into n equal segments, then

$$B_i = \left[p_{(i-1)/n} \quad p_{i/n} \quad \frac{1}{n}p'_{(i-1)/n} \quad \frac{1}{n}p'_{i/n}\right]$$

Composite Curves

- When two or more curves segments are joined together, they form a continuous composite curve



- Blending a new curve between two existing curve
- Let $B_1 = [P_1(0) \;\; P_1(1) \;\; P_1'(0) \;\; P_1'(1)]$
- and $B_3 = [P_3(0) \;\; P_3(1) \;\; P_3'(0) \;\; P_3'(1)]$
- End point must coincide
- So that $P_2(0) = P_1(1)$ and $P_2(1) = P_3(0)$

- Further, the tangent at each end of the new curve must match the tangent of adjoining curve.
- Magnitude may be different.
- Thus

$$P_2'(0) = a\frac{P_1'(1)}{|P_1'(1)|}$$

and

$$P_2'(1) = b\frac{P_3'(0)}{|P_3'(0)|}$$

Then

$$B_2 = \left[P_1(1) \quad P_3(0) \quad a\frac{P_1'(1)}{|P_1'(1)|} \quad b\frac{P_3'(0)}{|P_3'(0)|}\right]$$

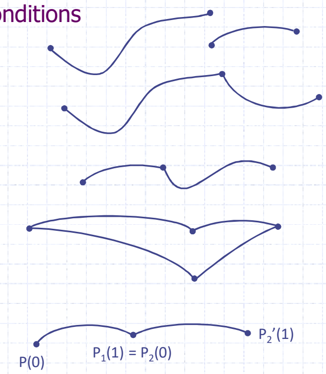
Where a and b are positive scale factor

- Expression for any PC curve smoothly and continuously blended with preceding and succeeding curves:

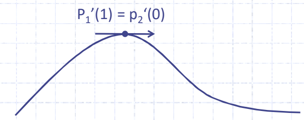
$$B_i = \left[P_{i-1}(1) \quad P_{i+1}(0) \quad a\frac{P_{i-1}'(1)}{|P_{i-1}'(1)|} \quad b\frac{P_{i+1}'(0)}{|P_{i+1}'(0)|}\right]$$

Continuity Conditions

- C^0 continuity
  - No gaps



- C^1 continuity between two curves required a common tangent line at their joining point.



- C^2 continuity also requires that two curves posses equal curvature at their joint.

$p_i(1) = p_{i+1}(0)$

$p_i'(1) = K_1 p_{i+1}'(0)$

$p_i''(1) = K_2 p_{i+1}''(0)$

Que. 1: Given a cubic hermite curve whose geometric coefficients are

$$B = [P_0 \quad P_1 \quad P_0' \quad P_1']$$

truncate the curve at u = 0.2 and u = 0.7 and reparameterize the remaining segment so that v ϵ [0 1]. Find the relationship between the geometric coefficients of truncated curve and those of original.

Que. 2: Given a cubic hermite curve whose geometric coefficients are

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 4 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

sub divide this curve into three segments with joints at u = 1/3 and u = 2/3. Reparameterize each segment over a unit interval and compute the three sets of geometric coefficients.

Que.: Given a cubic hermite curve whose geometric coefficients are

$$B = [P_0 \quad P_1 \quad P_0' \quad P_1']$$

truncate the curve at u = 0.2 and u = 0.7 and reparameterize the remaining segment so that v ϵ [0 1]. Find the relationship between the geometric coefficients of truncated curve and those of original.

$$P = UMB$$

$$B = [P_0 \quad P_1 \quad P_0' \quad P_1']$$

$$P(u) = [u^3 \quad u^2 \quad u \quad 1]MB$$

$$q_0 = P(0.2) = [0.2^3 \quad 0.2^2 \quad 0.2 \quad 1]MB$$

$$q_1 = P(0.7) = [0.7^3 \quad 0.7^2 \quad 0.7 \quad 1]MB$$

Tangent vectors are sensitive to the functional relationship between u and v, i.e. v = f(u), a linear relationship is required to preserve cubic form.

Thus v = au + b then dv = a.du

also $v_i = au_i + b$ and $v_j = au_j + b$ from this a and b can be found and relationship between tangent vectors

$$q_0' = \left(\frac{u_j - u_i}{v_j - v_i}\right) * P_0' \qquad q_0' = \left(\frac{0.7 - 0.2}{1 - 0}\right) * P_0'$$

$$q_1' = \left(\frac{u_j - u_i}{v_j - v_i}\right) * P_1' \qquad q_1' = \left(\frac{0.7 - 0.2}{1 - 0}\right) * P_1'$$

Now $q(v) = VM\boldsymbol{B}_1$

$$q(v) = [v^3 \quad v^2 \quad v \quad 1]M\boldsymbol{B}_1$$

$$\boldsymbol{B}_1 = [q_0 \quad q_1 \quad q_0' \quad q_1']$$

$$\boldsymbol{B}_1 = [P(0.2) \quad P(0.7) \quad 0.5P_0' \quad 0.5P_1']$$

Que.: Given a cubic hermite curve whose geometric coefficients are

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 4 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

sub divide this curve into three segments with joints at u = 1/3 and u = 2/3. Reparameterize each segment over a unit interval and compute the three sets of geometric coefficients.

- To subdivide a pc curve into n successive segments of arbitrary length and generate n new pc curves, then geometric coefficient or B matrix of ith segment are

$$B_i = [p_{i-1} \quad p_i \quad (u_i - u_{i-1})p'_{i-1} \quad (u_i - u_{i-1})p'_i]$$

If a curve is divided into n equal segments, then

$$B_i = \left[p_{(i-1)/n} \quad p_{i/n} \quad \frac{1}{n} p'_{(i-1)/n} \quad \frac{1}{n} p'_{i/n} \right]$$

$$P = UMB$$

$$P(u) = [u^3 \quad u^2 \quad u \quad 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_0' \\ P_1' \end{bmatrix}$$

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$$P(u) = [2u^3 - 3u^2 + 1 \quad -2u^3 + 3u^2 \quad u^3 - 2u^2 + u \quad u^3 - u^2] \begin{bmatrix} P_0 \\ P_1 \\ P_0' \\ P_1' \end{bmatrix}$$

$$P'(u) = [6u^2 - 6u \quad -6u^2 + 6u \quad 3u^2 - 4u + 1 \quad 3u^2 - 2u] \begin{bmatrix} P_0 \\ P_1 \\ P_0' \\ P_1' \end{bmatrix}$$

$$P'(u) = [0 \quad u^2 \quad u \quad 1] \begin{bmatrix} 0 & 0 & 0 & 0 \\ 6 & -6 & 3 & 3 \\ -6 & -6 & -4 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_0' \\ P_1' \end{bmatrix}$$

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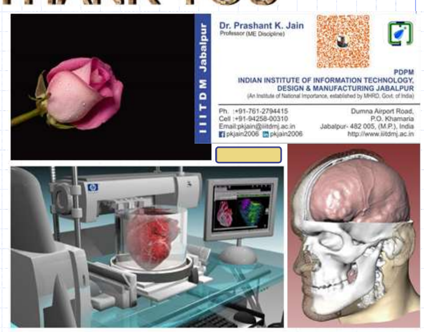

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