Computer Aided Geometric Design

Curves

B-Spline Curve

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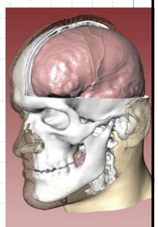

PDPM
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Parametric Representation of Free-form Curves

Interpolation Curves

- Parametric Cubic Curve

Approximation Curves

- Bezier Curve
- B-Spline Curve
- NURBS Curve

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B-Spline Curve

- Curve is defined by n+1 control points and the order (k) of the curve.
- The curve has an advantage that it has local propagation unlike global propagation properties of Bezier curve. i.e. Non global –each vertex B_i is associated with unique basis.
- Each vertex affects the shape only over a range of parameter value where its associated basis function is nonzero.
- B spline basis also allows the order of basis function and hence the degree of resulting curve can be changed without changing the number of vertices.
- The curve is made up of (n-k+2) segments.
- Only k control points affect any segment of the curve.

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B-Spline Curve

B spline curve is given by

$$p(t)=\sum_{i=1}^{n+1} B_i N_{i,k}(t) \qquad t_{\min}<t\le t_{\max} \qquad 2<k\le n+1$$

B_i- Position vector of n+1 defining vertices
$N_{i,k}$- Normalized B-Spline basis function

For i^{th} normalized B-Spline basis function of order k (degree k-1) the basis function $N_{i,k}(t)$ defined as: (cox de-boor formula)

$$N_{i,1}(t)=\begin{cases}1, if\ x_i\le t\le x_{i+1}\\ 0, otherwise\end{cases}$$

and

$$N_{i,k}(t)=\frac{(t-x_i)N_{i,k-1}(t)}{x_{i+k-1}-x_i}+\frac{(x_{i+k}-t)N_{i+1,k-1}(t)}{x_{i+k}-x_{i+1}}$$

X_i are elements of knot vector where $x_i \le x_{i+1}$
The function p(t) is polynomial of degree k-1 on each interval

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B-Spline Curve

Input to B-Spline Curve

1^{st} point, $p_0 = (x_0, y_0, z_0)$
2^{nd} point, $p_1 = (x_1, y_1, z_1)$
|
n^{th} point, $p_n = (x_n, y_n, z_n)$

Order of curve = k

Knot vector

$$N_{i,k}(t)=\frac{(t-x_i)N_{i,k-1}(t)}{x_{i+k-1}-x_i}+\frac{(x_{i+k}-t)N_{i+1,k-1}(t)}{x_{i+k}-x_{i+1}}$$

Equation for B-Spline curve of four defining vertices and order k=3 will be:

$$p(t) = N_{1,3}B_1 + N_{2,3}B_2 + N_{3,3}B_3 + N_{4,3}B_4$$

The basis function dependencies for $N_{1,3}$ are as:

$N_{1,3}$ $N_{2,3}$ $N_{3,3}$ $N_{4,3}$

$N_{1,2}$ $N_{2,2}$ $N_{3,2}$ $N_{4,2}$ $N_{5,2}$

$N_{1,1}$ $N_{2,1}$ $N_{3,1}$ $N_{4,1}$ $N_{5,1}$ $N_{6,1}$

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B-Spline Curve

- Each basis function is positive or zero for all parameter values, i.e., $N_{i,k} \ge 0$
- The sum of the B-spline basis functions for any parameter value t is one, i.e., $\sum_{i=1}^{n+1} N_{i,k}(t) = 1$
- Except for k=1, each basis has precisely one maximum.
- B-spline curve of order k (degree k-1) is c^{k-2} continuous everywhere.
- The maximum order of B-spline curve is equal to the number of defining polygon vertices.
- B-spline curve exhibits the variation diminishing property.
- B-spline curve generally follows the shape of the defining polygon.
- P(t) and its derivatives of order 1,2,-------k-2, are all contains over the entire curve.

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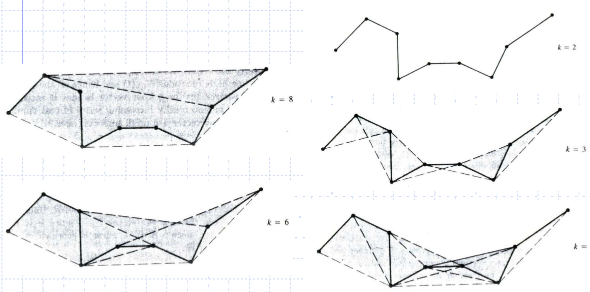
B-Spline Curve

- B-spline curve lies within the union of convex hulls formed by k successive defining polygon vertices.
- *Convex hull property of B-spline curves is stronger than Bezier Curves.*



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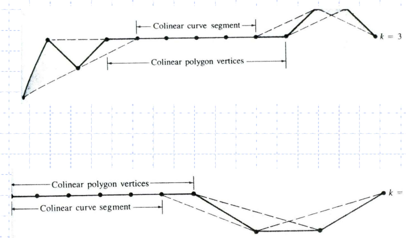
Linear Segment in B-Spline Curve



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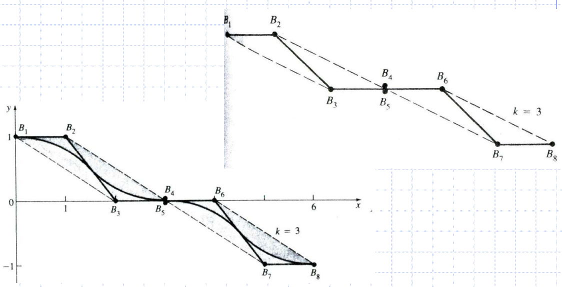
B-Spline Curve

If at least k-1 coincident defining polygon vertices occur i.e .,
$B_i = B_{i+1}$ ………………..B_{i+k-2} then convex hull of B_i to B_{i+k-2} is the vertex itself,
Hence the resulting B- spline curve must pass through the vertex Bi.



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B-Spline Curve: Knot Vectors

Fundamentally three types of knot vector are used:
Uniform, Open uniform, Non-uniform

In a **uniform knot vector**, knot values are uniformly / evenly spaced

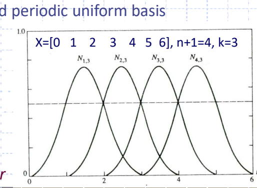
[0 1 2 3 4]
[-0.2 -0.1 0 0.1 0.2]

In practice uniform knot vectors begin at zero are incremented by 1 to some maximum value or normalized in the range between 0 to 1. i.e.,

[0 0.25 0.5 0.75 1.0]

For a given order *k* uniform knot vector yield periodic uniform basis function for which

$$N_{i,k}(t) = N_{i-1,k}(t-1) = N_{i+1,k}(t-1)$$



Each basis function is a translate of the other

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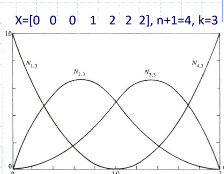
B-Spline Curve: Knot Vectors

An **open uniform knot vector** has multiplicity of knot values at the ends equal to the order K of B-Spline function.
Internal knot values are evenly spaced, e.g.

k=2 [0 0 1 2 3 4 4]
k=3 [0 0 0 1 2 3 3 3]
k=4 [0 0 0 0 1 2 2 2 2]



or for normalized increments

k=2 [0 0 ¼ ½ ¾ 1 1]
k=3 [0 0 0 ⅓ ⅔ 1 1 1]
k=4 [0 0 0 0 ½ 1 1 1 1]

Formally an open uniform knot vector is given by

$x_i = 0$ $1 \le i \le k$
$x_i = i-k$ $k+1 \le i \le n+1$
$x_i = n-k+2$ $n+2 \le i \le n+k+1$

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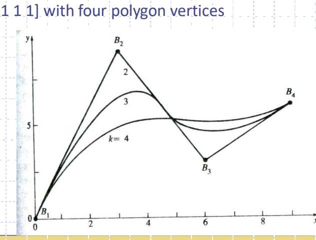
B-Spline Curve: Knot Vectors

The resulting **open uniform basis function** yields a curve that behave nearly like Bezier curves.

When number of defining polygon vertices are equal to the order of the B-Spline basis and a open uniform knot vector is used the B-Spline basis is reduces to the Bernstein basis, hence the resulting B spline curve is a Bezier curve.

In that case knot vector [0 0 0 0 1 1 1 1] with four polygon vertices results in a cubic Bezier Curve.



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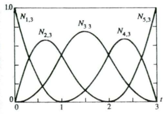
B-Spline Curve: Knot Vectors

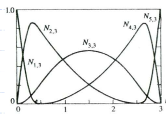
Non uniform knot vectors may have either unequally spaced and/or multiple internal knot values, they may be periodic or open. e. g.

[0 0 0 1 1 2 2 2]
[0 1 2 2 3 4]
[0 0.28 0.5 0.72 1]

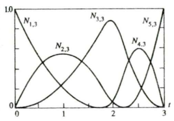
Basis function for n+1 = 5, k = 3



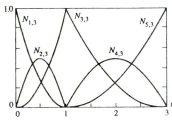
[x] = [0 0 0 1 2 3 3 3]



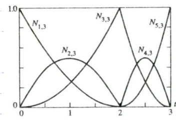
[x] = [0 0 0 0.4 2.6 3 3 3]



[x] = [0 0 0 1.8 2.2 3 3 3]



[x] = [0 0 0 1 1 3 3 3]



[x] = [0 0 0 2 2 3 3 3]

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Calculating Periodic Basis Function

Calculate four third order (k=3) basis function

$$N_{(i,3)}(t), i = 1,2,3,4$$

Here n+1, the number of basis function is 4, and curve is

$$p(t) = N_{1,3}B_1 + N_{2,3}B_2 + N_{3,3}B_3 + N_{4,3}B_4$$

The basis function dependencies for $N_{i,3}$ are as:

$N_{1,3}$ $N_{2,3}$ $N_{3,3}$ $N_{4,3}$

$N_{1,2}$ $N_{2,2}$ $N_{3,2}$ $N_{4,2}$ $N_{5,2}$

$N_{1,1}$ $N_{2,1}$ $N_{3,1}$ $N_{4,1}$ $N_{5,1}$ $N_{6,1}$

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Calculating Periodic Basis Function

$$N_{(i,1)}(t), i = \begin{cases} 1, if\ x_i \le t < x_{i+1} \\ 0, otherwise \end{cases}$$

Now knot vector range needed:

$$N_{i,k}(t) = \frac{(t - x_i)N_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t)N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}$$

Equation shows that calculation of $N_{6,1}$ requires knot values X_6 and X_7, while calculation of $N_{1,1}$ requires x_1 and x_2. Thus knot values form 0 to n+k are required.

Number of knot values is thus n+k+1. Hence the knot vector:

[x] = [0 1 2 3 4 5 6]

Where x_1=0 , x_2=1 - - - - -x_7=6 and Parameter and range is $0 \le t \le 6$.

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Calculating Periodic Basis Function

The basis function for various parameter ranges are:

$0 \le t < 1$

$$N_{1,1}(t) = 1, \quad N_{i,1}(t) = 0, \quad i \ne 1$$
$$N_{1,2}(t) = t, \quad N_{i,2}(t) = 0, \quad i \ne 1$$
$$N_{1,3}(t) = \frac{t^2}{2}, \quad N_{i,3}(t) = 0, \quad i \ne 1$$

$1 \le t < 2$

$$N_{2,1}(t) = 1, \quad N_{i,1}(t) = 0, \quad i \ne 2$$
$$N_{1,2}(t) = (2 - t), \quad N_{2,2}(t) = (t - 1) \quad N_{i,2}(t) = 0, \quad i \ne 1, 2$$
$$N_{1,3}(t) = \frac{t}{2}(2 - t) + \left(\frac{3 - t}{2}\right)(t-1),$$
$$N_{2,3}(t) = \frac{(t-1)^2}{2}, \quad N_{i,3}(t) = 0, \quad i \ne 1, 2, 3$$

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Calculating Periodic Basis Function

$2 \le t < 3$

$$N_{3,1}(t) = 1, \quad N_{i,1}(t) = 0, \quad i \ne 3$$
$$N_{2,2}(t) = (3 - t), \quad N_{3,2}(t) = (t - 2), \quad N_{i,2}(t) = 0, \quad i \ne 2,3$$
$$N_{1,3}(t) = \frac{(3-t)^2}{2}; \quad N_{2,3}(t) = \frac{(t-1)(3-t)}{2} + \frac{(4-t)(t-2)}{2};$$
$$N_{3,3}(t) = \frac{(t-2)^2}{2}, \quad N_{i,3}(t) = 0, \quad i \ne 1,2,3$$

$3 \le t < 4$

$$N_{4,1}(t) = 1, \quad N_{i,1}(t) = 0, \quad i \ne 4$$
$$N_{3,2}(t) = (4 - t), \quad N_{4,2}(t) = (t - 3), \quad N_{i,2}(t) = 0, \quad i \ne 3,4$$
$$N_{2,3}(t) = \frac{(4-t)^2}{2}; \quad N_{3,3}(t) = \frac{(t-2)(4-t)}{2} + \frac{(5-t)(t-3)}{2};$$
$$N_{4,3}(t) = \frac{(t-3)^2}{2}, \quad N_{i,3}(t) = 0, \quad i \ne 2,3,4$$

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Calculating Periodic Basis Function

$4 \le t < 5$

$$N_{5,1}(t) = 1, \quad N_{i,1}(t) = 0, \quad i \ne 5$$
$$N_{4,2}(t) = (5 - t), \quad N_{5,2}(t) = (t - 4), \quad N_{i,2}(t) = 0, \quad i \ne 4,5;$$
$$N_{3,3}(t) = \frac{(5-t)^2}{2};$$
$$N_{4,3}(t) = \frac{(t-3)(5-t)}{2} + \frac{(6-t)(t-4)}{2};$$
$$N_{i,3}(t) = 0; \quad i \ne 3,4$$

$5 \le t < 6$

$$N_{1,1}(t) = 1, \quad N_{i,1}(t) = 0, \quad i \ne 1$$
$$N_{1,2}(t) = t, \quad N_{i,2}(t) = 0, \quad i \ne 1$$
$$N_{1,3}(t) = \frac{t^2}{2}, \quad N_{i,3}(t) = 0, \quad i \ne 1$$

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Periodic Basis Functions

$N_{1,1}(t) = 1, \quad 0 \le t < 1$

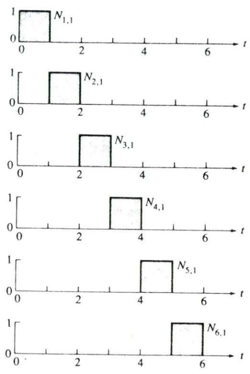
$N_{2,1}(t) = 1, \quad 1 \le t < 2$

$N_{3,1}(t) = 1, \quad 2 \le t < 3$

$N_{4,1}(t) = 1, \quad 3 \le t < 4$

$N_{5,1}(t) = 1, \quad 4 \le t < 5$

$N_{6,1}(t) = 1, \quad 5 \le t < 6$



(a)

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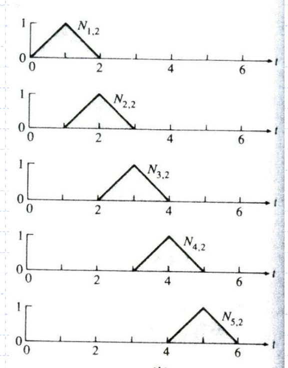
Periodic Basis Functions

$N_{1,2}(t) = t, \quad 0 \le t < 1$

$N_{1,2}(t) = (2 - t), \quad 1 \le t < 2$

$N_{2,2}(t) = (t - 1) \quad 1 \le t < 2$

$N_{2,2}(t) = (3 - t), \quad 2 \le t < 3$



(b)

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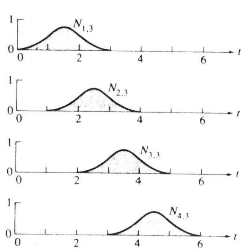
Periodic Basis Functions

$N_{1,3}(t) = \frac{t^2}{2}, \quad 0 \le t < 1$

$N_{1,3}(t) = \frac{t}{2}(2 - t) + \left(\frac{3-t}{2}\right)(t-1), \quad 1 \le t < 2$

$N_{1,3}(t) = \frac{(3 - t)^2}{2}, \quad 2 \le t < 3$



(c)

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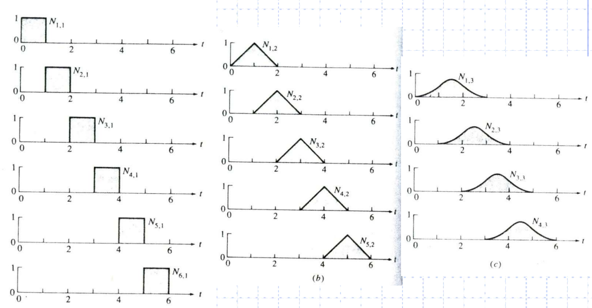
Periodic Basis Functions



(a) (b) (c)

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Calculating Open Uniform Basis Function

Calculate four (n=3) third order basis function (k=3) basis function $N_{i,3}$ i= 1,2,3,4. with an open knot vector.

Open knot vector with integer intervals between internal knot values is

$x_i = 0 \qquad 1 \le i \le k$

$x_i = i-k \qquad k+1 = 1 \le i \le n+1$

$x_i = n-k+2 \qquad n+2 \le i \le n+k+1$

Parameter range is $0 \le t \le n-k+2$, i.e., zero to maximum knot value and number of knot values n+k+1

Therefore the knot vector is:

[x] = [0 0 0 1 2 2 2]

Where $x_1=0$, $x_2=0$ - - - - $-x_7=2$ and Parameter and range is $0 \le t \le 2$.

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Calculating Open Uniform Basis Function

The basis function for various parameter ranges are:

$0 \le t < 1$

$N_{3,1}(t) = 1 \qquad N_{i,1}(t)=0 \qquad i\neq 3$

$N_{2,2}(t) = 1-t \qquad N_{3,2}(t)=t \qquad N_{i,2}(t)=0 \qquad i\neq 2,3$

$N_{1,3}(t) = (1-t)^2 \qquad N_{2,3}(t) = t(1-t)+(2-t)t/2$

$N_{3,3}(t) = t^2/2 \qquad N_{i,3}(t) = 0 \qquad i\neq 1,2,3$

$1 \le t < 2$

$N_{4,1}(t) = 1 \qquad N_{i,1}(t) = 0 \qquad i\neq 4$

$N_{3,2}(t) = (2-t) \qquad N_{4,2}(t) = (t-1) \qquad N_{i,2}(t) = 0 \qquad i \neq 3,4$

$N_{2,3}(t) = (2-t)^2/2 \qquad N_{3,3}(t) = t(2-t)/2+(2-t)(t-1)$

$N_{4,3}(t) = (t-1)^2 \qquad N_{i,3}(t) = 0 \qquad i\neq 2,3,4$

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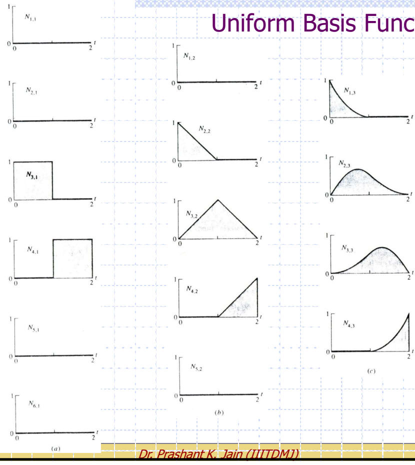
Uniform Basis Functions



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$$N_{i,1}(t) = \begin{cases} 1 & \text{if } x_i \le t \le x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

$$N_{i,k}(t) = \frac{(t - x_i)\, N_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t)\, N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}$$

$\underline{n=3}$ $\underline{k=3}$ $N_{i,3}(t)$ $i = 1, 2, 3, 4$ open knot vector

$x_i = 0$ $1 \le i < k$

$x_i = i-k$ $k+1 \le i < n+1$

$x_i = n-k+2$ $n+2 \le i \le n+k+1$

no. of knot values $n+1+k = 4+3 = 7$

$X = [0\ 0\ 0\ 1\ 2\ 2\ 2]$ $0 \le t \le 2$

$P(t) = N_{1,3} P_1 + N_{2,3} P_2 + N_{3,3} P_3 + N_{4,3} P_4$

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$$N_{i,k}(t) = \frac{(t - x_i)\, N_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t)\, N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}$$

$\underline{0 \le t < 1}$

$N_{3,1}(t) = 1$ $N_{i,1}(t) = 0$ $i \ne 3$

$N_{1,3}(t) = \frac{(t-0) N_{1,2}}{0-0} + \frac{(1-t) N_{2,2}(t)}{1-0} = (1-t)^2$

$N_{1,2} = \frac{(t-0) N_{1,1}}{0} + \frac{(0-t) N_{2,1}}{0-0} = 0$

$N_{2,2} = \frac{(t-0) N_{2,1}}{0-0} + \frac{(1-t) N_{3,1}}{1-0} = (1-t)$

$N_{2,3}(t) = \frac{(t-0) N_{2,2}}{1-0} + \frac{(2-t) N_{3,2}}{2-0} = t(1-t) + \frac{(2-t)}{2} t$

$N_{3,2} = \frac{(t-0) N_{3,1}}{1-0} + \frac{(2-t) N_{4,1}}{2-0} = t$

$N_{3,3} = \frac{(t-0) N_{3,2}}{2-0} + \frac{(2-t) N_{4,2}}{2-\phi} = \frac{t \cdot t}{2} = \frac{t^2}{2}$

$N_{4,2} = \frac{(t-1) N_{4,1}}{2-\phi} + \frac{(2-t) N_{5,1}}{2-\cancel{2}} = 0$

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$$N_{i,k}(t) = \frac{(t - x_i)\, N_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t)\, N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}$$

$\underline{1 \le t\ 2}$

$N_{4,1}(t) = 1$ $N_{i,1}(t) = 0$ $i \ne 4$

$N_{1,3}(t) = \frac{(t-0) N_{1,2}}{0-0} = 0$

$N_{2,3}(t) = \frac{(t-0) N_{2,2}}{1-0} + \frac{(2-t) N_{3,2}}{2-0} = 0 + \frac{(2-t)}{2} \times (2-t) = \frac{(2-t)^2}{2}$

$N_{2,2} = \frac{(t-0) N_{2,1}}{1-0} + \frac{(1-t) N_{3,1}}{1-0} = 0+0 = 0$

$N_{3,2} = \frac{(t-0) N_{3,1}}{1-0} + \frac{(2-t) N_{4,1}}{2-1} = 0 + \frac{2-t}{1}$

$N_{3,3}(t) = \frac{(t-0) N_{3,2}}{2-0} + \frac{(2-t) N_{4,2}}{2-1} = \frac{t}{2}(2-t) + (2-t)(t-1)$

$N_{4,2}(t) = \frac{(t-1) N_{4,1}}{2-1} + \frac{(2-t) N_{5,1}}{2-2} = (t-1) + 0$

$N_{4,3}(t) = \frac{(t-1) N_{4,2}}{2-1} + \frac{(2-t) N_{5,2}}{2-2} = \frac{(t-1)(t-1)}{1} = (t-1)^2$

$N_{5,2}(t) = \frac{(t-2) N_{5,1}}{2-2} + \frac{(2-t) N_{6,1}}{2-2} = 0$

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Calculating Open Uniform Basis Function

Calculate five (n=4) third order basis function (k=3) basis function $N_{i,3}$ i= 1,2,3,4,5. with an open knot vector.

Open knot vector with integer intervals between internal knot values is

$x_i = 0$ $1 \le i \le k$

$x_i = i-k$ $k+1 = 1 \le i \le n+1$

$x_i = n-k+2$ $n+2 \le i \le n+k+1$

Parameter range is $0 \le t \le n-k+2$, i.e., zero to maximum knot value and number of knot values n+k+1

Therefore the knot vector is:

[x] = [0 0 0 1 2 3 3 3]

Where $x_1=0$, $x_2=0$ - - - - $-x_8=3$ and Parameter and range is $0 \le t \le 3$.

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Calculating Non-uniform Basis Function

Calculate the five (n+1 = 5) third – order (k = 3) basis functions $N_{i,3}(t)$, i=1, 2, 3, 4, 5 using the knot vector [X] = [0 0 0 1 1 3 3 3] which contains an interior repeated knot value.

$$N_{i,1}(t) = \begin{cases} 1, \text{ if } x_i \le t \le x_{i+1} \\ 0, \text{ otherwise} \end{cases}$$

and

$$N_{i,k}(t) = \frac{(t - x_i)\, N_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t)\, N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}$$

0 ≤ t < 1

$N_{3,1}(t) = 1$; $N_{i,1}(t) = 0$, $i \ne 3$

$N_{2,2}(t) = 1-t$; $N_{3,2}(t) = t$; $N_{i,2}(t) = 0$, $i \ne 2,3$

$N_{1,3}(t) = (1-t)^2$; $N_{2,3}(t) = t(1-t) + (1-t)t = 2t(1-t)$;

$N_{3,3}(t) = t^2$; $N_{i,3}(t) = 0$, $i \ne 1,2,3$

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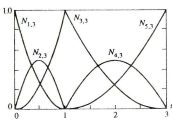
Calculating Non-uniform Basis Function



$[x] = [0\ 0\ 0\ 1\ 1\ 3\ 3\ 3]$

$1 \le t < 1$

$N_{i,1}(t) = 0$, all i

$N_{i,2}(t) = 0$, all i

$N_{i,3}(t) = 0$, all i

Notice specifically that as a consequence of the multiple knot value, $N_{4,1}(t) = 0$ for all t.

$1 \le t < 3$

$N_{5,1}(t) = 1$; $N_{i,1}(t) = 0$, $i \ne 5$

$N_{4,2}(t) = (3-t)/2$; $N_{5,2}(t) = (t-1)/2$; $N_{i,2}(t) = 0$, $i \ne 4,5$

$N_{3,3}(t) = (3-t)^2/4$;

$N_{4,3}(t) = (t-1)(3-t)/4 + (3-t)(t-1)/4 = (3-t)(t-1)/2$;

$N_{5,3}(t) = (t-1)^2/4$; $N_{i,3}(t) = 0$, $i \ne 3, 4, 5$

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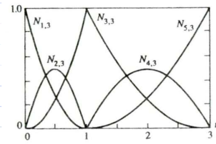
Calculating Non-uniform Basis Function



$[x] = [0\ 0\ 0\ 1\ 1\ 3\ 3\ 3]$

Notice that for each value of t the $\sum N_{i,k}(t) = 1.0$. For example, with

$0 \le t < 1$

$$\sum_{i=1}^{5} N_{i,3}(t) = (1-t)^2 + 2t(1-t) + t^2$$

$$= 1-2t+t^2+2t-2t^2+t^2 = 1$$

Similarly, for **$1 \le t < 3$**

$$\sum_{i=1}^{5} N_{i,3}(t) = \frac{1}{4}[(3-t)^2 + 2(3-t)(t-1) + (t-1)^2]$$

$$= \tfrac{1}{4}[9-6t+t^2-6t+8t-2t^2+1-2t+t^2] = 4/4 = 1$$

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Flexibility in B-Spline Curves

The shape of B-spline curve can be controlled by :

- Changing type of knot vector and hence basis function
- Changing order (k) of the basis function
- Using multiple polygon vertices
- Using multiple knot value in the knot vector
- Changing number and position of defining polygon vertices

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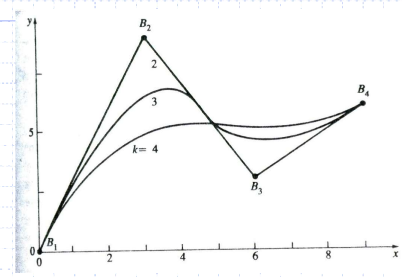
Effect of varying order on open B-spline curve



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Effect of multiple or coincident vertices

Multiple vertices at B_2, k=4

For k=4

1= [0 0 0 0 1 1 1 1]

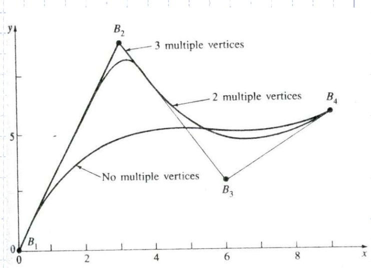
$[B_1\ B_2\ B_3\ B_4]$

2= [0 0 0 0 1 2 2 2 2]

$[B_1\ B_2\ B_2\ B_3\ B_4]$

3= [0 0 0 0 1 2 3 3 3 3]

$[B_1\ B_2\ B_2\ B_2\ B_3\ B_4]$



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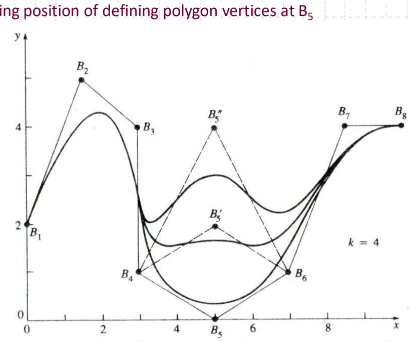
Local control of B-Spline Curves

Changing position of defining polygon vertices at B_5



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Derivatives of B spline curve at any point

$$p(t) = \sum_{i=1}^{n+1} B_i N_{i,k}(t)$$

$$p'(t) = \sum_{i=1}^{n+1} B_i N'_{i,k}(t)$$

$$N'_{i,k}(t) = \frac{N_{i,k-1}(t) + (t - x_i) N'_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t) N'_{i+1,k-1}(t) - N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}$$

$$N'_{i,1}(t) = 0$$

for all t and for k=2

$$N'_{i,2}(t) = \frac{N_{i,1}(t)}{x_{i+k-1} - x_i} - \frac{N'_{i+1,1}(t)}{x_{i+k} - x_{i+1}}$$

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Derivatives of B spline curve at any point

$$p''(t) = \sum_{i=1}^{n+1} B_i N''_{i,k}(t)$$

$$N''_{i,k}(t) = \frac{2N_{i,k-1}(t) + (t - x_i) N''_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t) N''_{i+1,k-1}(t) - 2N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}$$

$N''_{i,1}(t) = 0$ and $N''_{i,2}(t) = 0$ For all t.

For k=3

$$N''_{i,3}(t) = 2\left(\frac{N'_{i,2}(t)}{x_{i+k-1} - x_i} - \frac{N'_{i+1,2}(t)}{x_{i+k} - x_{i+1}}\right)$$

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Numericals: Calculating an open B-Spline Curve

Given B_1 [1 1] , B_2 [2,3], B_3[4,3] and B_4[3,1] the vertices of a polygon. Calculate both second and fourth – order B – Spline curves.

For k= 2 the open knot vector is [0 0 1 2 3 3]

Where $x_1 = 0$, $x_2 = 0$, ……, $x_6 = 3$. The parameter range is $0 \le t \le 3$. The curve is composed of three linear (k-1 =1) segments. For $0 \le t \le 3$ the basis functions are :

$0 \le t < 1$

$N_{2,1}(t) = 1$; $N_{i,1}(t) = 0$, $i \ne 2$

$N_{1,2}(t) = 1-t$; $N_{2,2}(t) = t$; $N_{i,2}(t) = 0$, $i \ne 1,2$

$1 \le t < 2$

$N_{3,1}(t) = 1$; $N_{i,1}(t) = 0$, $i \ne 2$

$N_{2,2}(t) = 2-t$; $N_{3,2}(t) = (t-1)$; $N_{i,2}(t) = 0$, $i \ne 2,3$

$2 \le t < 3$

$N_{4,1}(t) = 1$; $N_{i,1}(t) = 0$, $i \ne 4$

$N_{3,2}(t) = (3-t)$; $N_{4,2}(t) = (t-2)$; $N_{i,2}(t) = 0$, $i \ne 3,4$

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Numericals: Calculating an open B-Spline Curve

Using equation $p(t) = \sum_{i=1}^{n+1} B_i N_{i,k}(t)$ $\quad t_{\min} < t \le t_{\max}$, $2 < k \le n+1$

The parametric B-Spline curve is

$P(t) = B_1 N_{1,2}(t) + B_2 N_{2,2}(t) + B_3 N_{3,2}(t) + B_4 N_{4,2}(t)$

For each of these intervals the curve is given by

$P(t) = (1-t)B_1 + tB_2 = B_1 + (B_2 - B_1)t$		$0 \le t < 1$
$P(t) = (2-t)B_2 + (t-1)B_3 = B_2 + (B_3 - B_2)t$		$1 \le t < 2$
$P(t) = (3-t)B_3 + (t-2)B_4 = B_3 + (B_4 - B_3)t$		$2 \le t < 3$

In each case the result is the equation of the parametric straight line for the polygon span, i.e., the 'curve' is the defining polygon.

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Numericals: Calculating an open B-Spline Curve

For k=4 the order of the curve is equal to the number of defining polygon vertices.

Thus the B-spline curve reduces to a Bezier curve. The knot vector with $t_{max} = n-k+2 = 3-4+2 = 1$ is [0 0 0 0 1 1 1 1].

The basis functions are:

$0 \le t < 1$

$N_{4,1}(t) = 1$; $N_{i,1}(t) = 0$, $i \ne 4$

$N_{3,2}(t) = (1-t)$; $N_{4,2}(t) = t$; $N_{i,2}(t) = 0$, $i \ne 3,4$

$N_{2,3}(t) = (1-t)^2$; $N_{3,3}(t) = 2t(1-t)$;

$N_{4,3}(t) = t^2$; $N_{i,3}(t) = 0$, $I \ne 2,3,4$

$N_{1,4}(t) = (1-t)^3$; **$N_{2,4}(t) = t(1-t)^2 + 2t(1-t)^2 = 3t(1-t)^2$;**

$N_{3,4}(t) = 2t^2(1-t) + (1-t)t^2 = 3t^2(1-t)$; **$N_{4,4}(t) = t^3$;**

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Numericals: Calculating an open B-Spline Curve

Using equation $p(t) = \sum_{i=1}^{n+1} B_i N_{i,k}(t)$ $\quad t\max_{\min}$

The parametric B-Spline curve is

$P(t) = B_1 N_{1,4}(t) + B_2 N_{2,4}(t) + B_3 N_{3,3}(t) + B_4 N_{4,4}(t)$

$P(t) = (1-t)^3 B_1 + 3t(1-t)^2 B_2 + 3t^2(1-t) B_3 + t^3 B_4$

Thus, at t=0 $\quad P(0) = B_1$

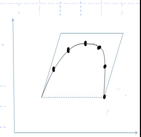
Thus, at t=1 $\quad P(1) = B_4$

In this case B Spline curve reduces to Bezier Curve

and at t= ½ $\quad P(\frac{1}{2}) = \frac{1}{8}B_1 + \frac{3}{8}B_2 + \frac{3}{8}B_3 + \frac{1}{8}B_4$

and $\quad P(\frac{1}{2}) = \frac{1}{8}[1 \;\; 1] + \frac{3}{8}[2 \;\; 3] + \frac{3}{8}[4 \;\; 3] + \frac{1}{8}[3 \;\; 1]$

= [11/4 5/2]



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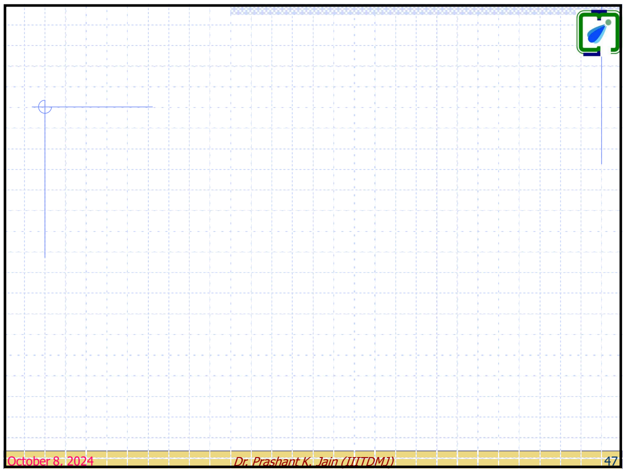
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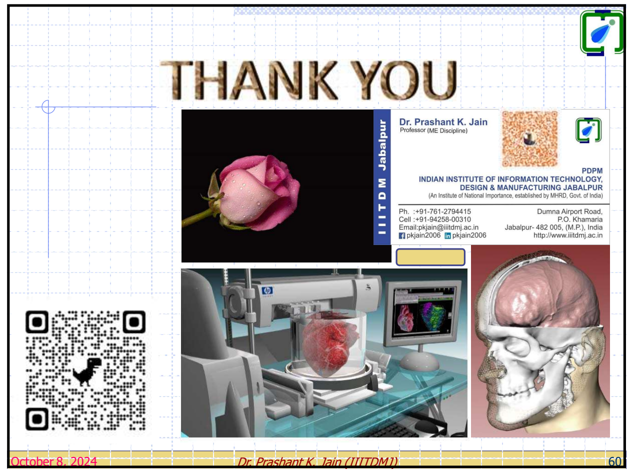
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