Computer Aided Geometric Design

Surfaces

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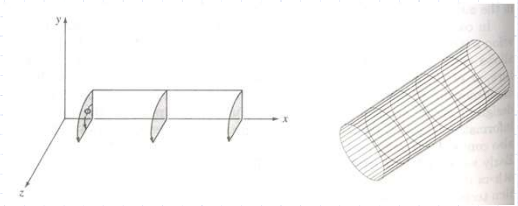
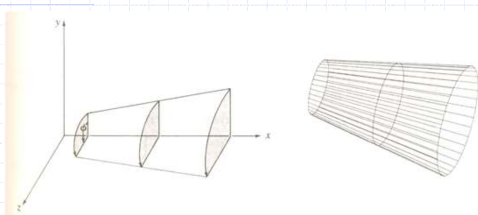
(An Institute of National Importance (INI) established by MHRD, Govt. of INDIA)

Surface of revolution

- Simplest method
- Revolve a two dimensional entity
  - Line or a plane curve about an axis



Conical Surface of revolution



Surface of revolution

For parametric equation of a point on a surface of revolution

the parametric equation of the entity to be rotated

$$P(t) = [x(t) \;\; y(t) \;\; z(t)] \qquad 0 \le t \le t_{\max}$$

function of the single parameter t.

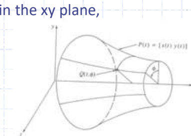
Rotation about an axis causes the location of the point to also be a function of the rotation angle Ø.

Thus, a point on a surface of revolution is specified by two parameters t and Ø.

It is bi-parametric function.

Rotation about the x-axis of an entity, initially lying in the xy plane, the surface equation is;

$$Q(t,\varphi) = [x(t) \;\; y(t)\cos\varphi \;\; y(t)\sin\varphi]$$

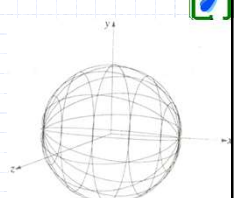


Sphere

Rotating plane curves also yields surfaces of revolution.

A sphere is obtained by rotating an origin-centered semicircle in the *xy* plane about the x-axis.

the parametric equation of the circle



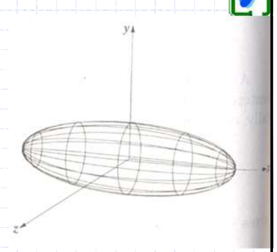
$$x = r\cos\theta \qquad 0 \le \theta \le \pi$$
$$y = r\sin\theta$$

the parametric equation of the sphere is

$$Q(\theta,\varphi) = [x(\theta) \;\; y(\theta)\cos\varphi \;\; y(\theta)\sin\varphi] \qquad 0 \le \theta \le \pi$$
$$= [r\cos\theta \;\; r\sin\theta\cos\varphi \;\; r\sin\theta\sin\varphi] \quad 0 \le \varphi \le 2\pi$$

Ellipsoid

An ellipsoid of revolution is obtained by rotating parametric equation of an origin centered semi ellipse in the *xy* plane.



the parametric equation of the semi ellipse

$$x = a\cos\theta \;\; 0 \le \theta \le \pi$$
$$y = b\sin\theta$$

parametric equation for any point on the ellipsoid of revolution is

$$Q(\theta,\varphi) = [a\cos\theta \;\; b\sin\theta\cos\varphi \;\; b\sin\theta\sin\varphi] \quad 0 \le \theta \le \pi$$
$$0 \le \varphi \le 2\pi$$

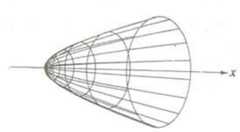
Paraboloid

A paraboloid of revolution is obtained by rotating the parametric parabola

$$x = a\theta^2 \qquad 0 \le \theta \le \theta_{\max}$$
$$y = 2a\theta$$

about the x-axis. The parametric surface is given by

$$Q(\theta,\varphi) = [a\theta^2 \quad 2a\theta\cos\varphi \quad 2a\theta\sin\varphi] \qquad 0 \le \theta \le \theta_{\max},\ 0 \le \varphi \le 2\pi$$



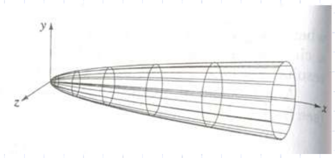
Hyperboloid

A hyperboloid of revolution is obtained by rotating the parametric hyperbola

$$x = a\sec\theta \qquad 0 \le \theta \le \theta_{\max}$$
$$y = b\tan\theta$$

about the x-axis. The parametric surface is given by

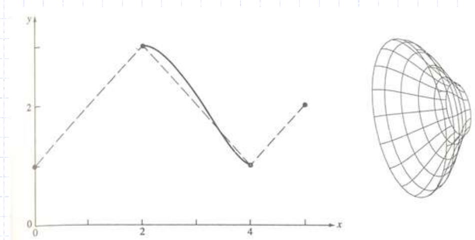
$$Q(\theta,\varphi) = [a\sec\theta \quad b\tan\theta\cos\varphi \quad b\tan\theta\sin\varphi] \qquad 0 \le \theta \le \theta_{\max},\ 0 \le \varphi \le 2\pi$$



Surface of Revolution

Any parametric curve can be used to create a surface of revolution.

Like cubic spline, Bezier and B-spline curves.



Surface of Revolution

in matrix form a parametric space curve is given by

P(t)=[T] [N] [G]

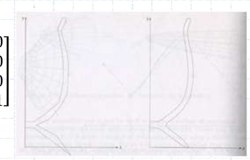
where [T], [N] and [G] are parameter, blending function and geometry matrices, respectively.

The general form of the matrix equation for a surface of revolution is

Q(t, ø)=[T] [N] [G] [S]

where [S] represents the contribution due to rotation about an axis by the angle ø. For the specific case of rotation about the x-axis

$$[S] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Derivatives: Surface of Revolution

Formally differentiating yields the parametric derivatives for a surface of revolution.

Specifically, the derivative in the axial direction is

Q_t(t, ø)=[T'] [N] [G] [S]

and in the radial direction

$Q_ø$(t, ø)=[T] [N] [G] [S']

where the prime denotes appropriate differentiation.

The surface normal is given by the cross product of the parametric derivatives, i.e.,

n=Q_t X $Q_ø$

Finding point on surface

Consider the line segment with end points P_1[1 1 0] and P_2[6 2 0] lying in the *xy* plane. Rotating the line about the x-axis yields a conical surface. Determine the point on this surface at t=0.5, $\varphi = \pi/3(60^0)$

The parametric equation for the line segment from P_1 to P_2 is

$$P(t) = [x(t) \quad y(t) \quad z(t)]$$
$$P(t) = P_1 + (P_2 - P_1)t \qquad 0 \le t \le 1$$

With cartesian components
$$x(t) = x_1 + (x_2 - x_1)t = 1 + 5t$$
$$y(t) = y_1 + (y_2 - y_1)t = 1 + t$$
$$z(t) = z_1 + (z_2 - z_1)t = 0$$

Contd…

Finding point on surface

The point $Q(1/2, \pi/3)$ on the surface of revolution is

$$Q(1/2, \pi/3) = [1+5t \quad (1+t)\cos\varphi \quad (1+t)\sin\varphi]$$

$$= \begin{bmatrix} \frac{7}{2} & \frac{3}{2}\cos\left(\frac{\pi}{3}\right) & \frac{3}{2}\sin\left(\frac{\pi}{3}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{2} & \frac{3}{4} & \frac{3\sqrt{3}}{4} \end{bmatrix}$$

$$= [3.5 \quad 0.75 \quad 1.3]$$

Sweep Surface

- A three dimensional surface is obtained by traversing an entity, e.g., a line, polygon or curve, along a path in space called sweep surfaces.
- Frequently used in geometric modeling.
- Consider the position vector P[x y z 1] swept along the path represented by the sweep transformation [T(s)]. The position vector Q(s) representing the resulting curve is given by

$$Q(s) = P[T(s)] \qquad s_1 \le s \le s_2$$

- The transformation [T(s)] determines the shape of the curve. For example, if the path is straight line of length n parallel to the z-axis, then

$$[T(s)] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & n_s & 1 \end{bmatrix} \qquad 0 \le s \le 1$$

Sweep Surface

The simplest sweep surface is obtained by traversing a line segment along a path.

Parametric equation of a line segment is

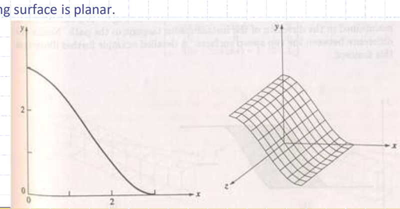
$$P(t) = P_1 + (P_2 - P_1)t \qquad 0 \le t \le 1$$

The corresponding sweep surface is given by

$$Q(t,s) = P(t)[T(s)] \qquad 0 \le t \le 1, \qquad s_1 \le s \le s_2$$

[T(s)] is the sweep transformation.

If the sweep transformation contains only translations and/ or local or overall scalings, the resulting surface is planar.

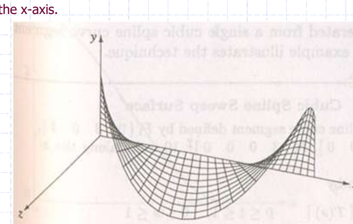


Sweep Surface

If the sweep transformation contains rotations, the resulting surface is non-planar.

The helical sweep surface obtained by simultaneously translating along and rotating about the x-axis a line originally parallel to the y-axis.

- Consider the line segment in the xy plane and parallel to the y-axis defined by end points $P_1[0 \ 0 \ 0]$ and $P_2[0 \ 3 \ 0]$. Determine the point at t=0.5, s=0.5 on the sweep surface generated by simultaneously translating the line 10 units along the x-axis and rotating it through 2π about the x-axis.



Here the sweep transformation matrix is a translation followed by a rotation given by

$$[T(s)] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\pi s) & \sin(2\pi s) & 0 \\ 0 & -\sin(2\pi s) & \cos(2\pi s) & 0 \\ ls & 0 & 0 & 1 \end{bmatrix}$$

The parametric equation of the line segment is

$P(t)=P_1+(P_2- P_1)t$

= [0 0 0 1]+[0-0 3-0 0-0 1-1]t

= [0 3t 0 1]

The sweep surface is given by

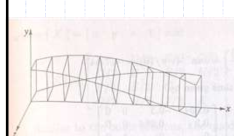
Q(t, s)=[P(t)] [T(s)]

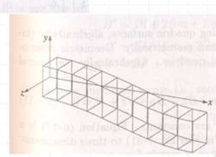
$$= [0 \ 3t \ 0 \ 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\pi s) & \sin(2\pi s) & 0 \\ 0 & -\sin(2\pi s) & \cos(2\pi s) & 0 \\ ls & 0 & 0 & 1 \end{bmatrix}$$

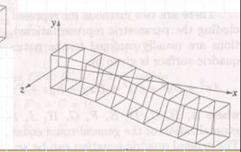
$$Q(0.5, 0.5) = [0 \ 1.5 \ 0 \ 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix}$$

$$= [5 \ -1.5 \ 0 \ 1]$$

Sweep Surface







Sweep Surface

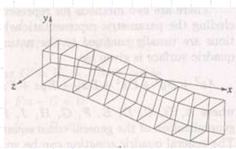
Sweep the planar square defined by vertices P_1 [0 -1 1], P_2 [0 -1 -1], P_3 [0 1 -1], P_4 [0 1 1] along the path x=10s, y=cos(πs)-1 while maintaining the normal to the polygon in the instantaneous direction of the tangent to the path

The instantaneous direction of the path tangent is [10 -πsin(πs) 0]. The rotation angle about the z-axis to align the polygon normal with the tangent to the path is thus

$$\psi = \tan^{-1}\left(\frac{-\pi\sin(\pi s)}{10}\right)$$

The sweep transformation is thus

$$[T(s)] = \begin{bmatrix} \cos\psi & \sin\psi & 0 & 0 \\ -\sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 10s & \cos(\pi s)-1 & 0 & 1 \end{bmatrix}$$



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Sweep Surface

At s=0.5 the rotation angle is

$$\psi = \tan^{-1}\left(\frac{-\pi\sin\left(\frac{\pi}{2}\right)}{10}\right)$$

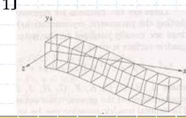
$$= \tan^{-1}\left(\frac{-\pi}{10}\right)$$

$$= -17.44^o$$

The polygonal square at s=0.5 is thus given by

$$Q = \begin{bmatrix} 0 & -1 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.954 & -0.3 & 0 & 0 \\ 0.3 & 0.954 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4.7 & -1.954 & 1 & 1 \\ 4.7 & -1.954 & -1 & 1 \\ 5.3 & -0.46 & -1 & 1 \\ 5.3 & -0.46 & 1 & 1 \end{bmatrix}$$



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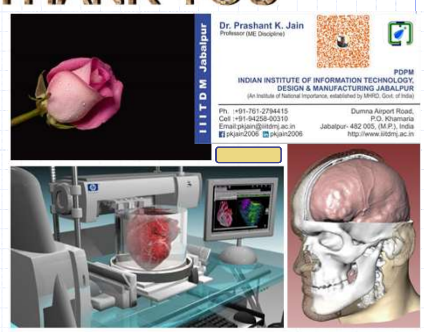


Piecewise surface representation

- Analytical description does not exist
  - Automobile bodies
  - Aircraft fuselages and wings
  - Ship hulls
  - Sculptures
  - Bottles
  - Shoes
- Surface represented in piece wise fashion
- A vector valued parametric representation is used
  - Axis independent
  - Avoids infinite slope values
  - Allows unambiguous representation of multi-valued surfaces
  - Facilitates representation in homogeneous coordinates
  - Compatible with use of 3D homogeneous coordinate transformations

Piecewise surface representation

Consider spherical surface

Specific curve on surface defined by plane intersecting sphere

Consider intersection of unit sphere and a plane defined by the surface equation z=cosϕ_1=a_1=constant

The resulting curve is a parallel of latitude.

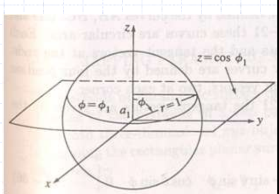
The parametric equation for a unit sphere is

$$x^2 + y^2 + z^2 = 1.0$$

Thus

$$x^2 + y^2 = 1 - {a_1}^2$$

Defines the intersection



Piecewise surface representation

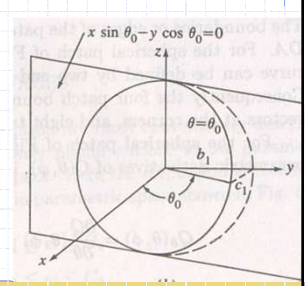
- The plane $\theta = \theta_0$=constant is defined by

$$x\sin\theta_0 - y\cos\theta_0 = 0$$

$$c_1 x - b_1 y = 0$$

- The intersection of this plane and a sphere yields a meridian of longitude. Solving these equations simultaneously yields the resulting curve; i.e.,

$$y^2\left[\left(\frac{b_1}{c_1}\right)^2 + 1\right] + z^2 = 1.0$$



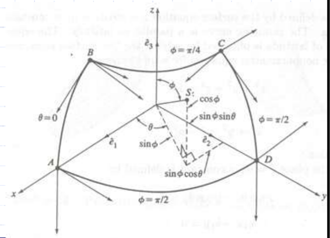
Spherical Surface Patch

- The boundaries of a spherical surface patch on a unit sphere can be formed by four planes, two parallels of latitude and two meridians of longitude, intersecting the sphere.
- The vector valued parametric equation for the resulting surface patch Q(θ, ø) is

$Q(\theta,\varphi) = [\cos\theta\sin\varphi \quad \sin\theta\sin\varphi \quad \cos\varphi] \quad \theta_1 \le \theta \le \theta_2, \quad \varphi_1 \le \varphi \le \varphi_2$

- The surface patch is the locus of a point in three-dimensional space which moves with two degrees of freedom controlled by the two parameter variables θ and ø, i.e., it is a biparametric

$0 \le \theta \le \frac{\pi}{2} and \frac{\pi}{4} \le \varphi \le \frac{\pi}{2}$



Piecewise surface representation

- These curves are circular arcs.
- Can be defined by two end points and the tangent vectors at the ends.
- Four patch boundary curves are defined by the four position vectors at the corners, and eight tangent vectors, two at each corner.
- For the spherical patch, the tangent vectors are given by the parametric derivatives of Q(θ, ø), i.e.,

$$Q_\theta(\theta,\varphi) = \frac{\partial Q}{\partial\theta}(\theta,\varphi) = [-\sin\theta\sin\varphi \quad \cos\theta\sin\varphi \quad 0]$$

$$Q_\varphi(\theta,\varphi) = [\cos\theta\cos\varphi \quad \sin\theta\cos\varphi \quad -\sin\varphi]$$

Piecewise surface representation

- The shape of the interior of the surface near each corner is controlled by the twist vector or cross derivative at the corner. For the spherical surface patch, the cross derivative or twist vector is

$$Q_{\theta,\varphi}(\theta,\varphi) = \frac{\partial^2 Q}{\partial\theta\partial\varphi} = \frac{\partial^2 Q}{\partial\varphi\partial\theta} = [-\sin\theta\cos\varphi \quad \cos\theta\cos\varphi \quad 0]$$

- Consequently a quadrilateral surface patch can be completely described by
  - The 4 position vectors at the corners.
  - The 8 tangent vectors, two at each corner.
  - The 4 twist vectors at the corners.

Piecewise surface representation

- The normal to a surface patch at any point is given by the cross product of the parametric derivatives.
- Specifically for a spherical surface.

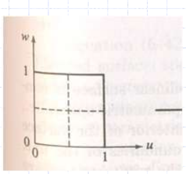
$$Q_\theta \times Q_\varphi = \begin{vmatrix} i & j & k \\ -\sin\theta\sin\varphi & \cos\theta\sin\varphi & 0 \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi \end{vmatrix}$$

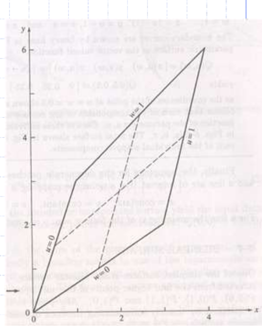
$$= [-\cos\theta\sin^2\varphi \quad \sin\theta\sin^2\varphi \quad -\sin\varphi\cos\varphi]$$

Mapping Parametric Surfaces

A surface in object space is represented by the functions that map this parametric surface into xyz object space, i.e.,

x=x(u, w)
y=y(u, w)
z=z(u, w)





Mapping Parametric Surfaces

Map the surface described by

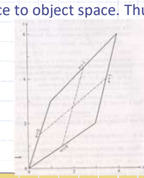
x=3u+w   0≤u ≤1
y=2u+3w+uw   0≤w ≤1
z=0

in parametric space into object space.

since z=constant=0, the surface in object space is also two-dimensional lying in the z=0 plane.

The boundaries of the surface in object space are defined by mapping the boundaries of the rectangle in parametric space to object space. Thus, for

u=0;   x=w, y=3w   and   y=3x
u=1   x=w+3, y=2(2w+1)   and   y=2(2x-5)
w=0   x=3u, y=2u   and   y=(2/3)x
w=1   x=3u+1, y=3u+3   and   y=x+2



Mapping Parametric Surfaces

Map the surface described by

$x(u, w)=(u-w)^2$ $0 \le u \le 1$

$y(u, w)=u-w^2$ $0 \le w \le 1$

$z(u, w)=uw$ in parametric space into object space.

Calculate the coordinates of the point at u=w=0.5 on the surface in object space.

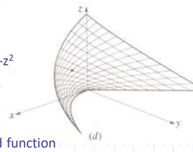
First determine the boundary curves

u=0; $x=w^2$, $y=-w^2$, z=0 and x=-y, z=0

u=1; $x=(1-w)^2$, $y=1-w^2$, z=w and $x=(1-z)^2$, $y=1-z^2$

w=0; $x=u^2$, y=u, z=0 and $x=y^2$, z=0

w=1; $x=(u-1)^2$, y=u-1, z=u and $x=y^2$, z=1+y



Writing the parametric surfaces as the vector valued function

Q(u, w)=[x(u, w) y(u, w) z(u, w)]

$=[(u-w)^2 \quad u-w^2 \quad uw]$

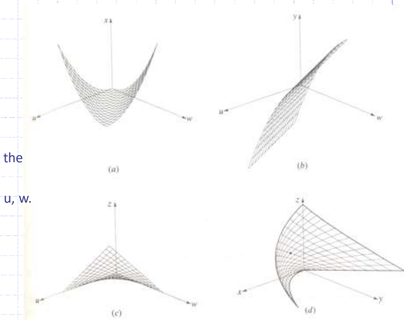
yields Q(0.5, 0.5)=[0 0.25 0.25]

as the coordinates of the point at u=w=0.5.

Mapping Parametric Surfaces

Three Dimensional Surface Mapping

(a) X component
(b) Y component
(c) Z component
(d) Complete



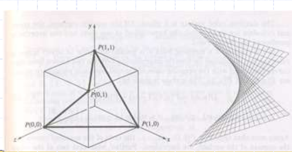
Each of the components of the surface in object space is a function of the parameters u, w.

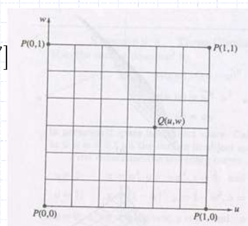
Bilinear surface

- Constructed from the four corner points of the unit square in parametric space, i.e., P(0, 0), P(0, 1), P(1, 1), P(1, 0).
- Any point in the interior of the surface is specified by linearly interpolating between opposite boundaries of the unit square.
- Any point in the interior of the parametric square is given by

  Q(u, w)=P(0, 0)(1-u)(1-w) + P(0, 1)(1-u)w + P(1, 0)u(1-w) + P(1, 1)uw
- In matrix form

$$Q(u,w) = [1-u \quad u]\begin{bmatrix} P(0,0) & P(0,1) \\ P(1,0) & P(1,1) \end{bmatrix}\begin{bmatrix} 1-w \\ w \end{bmatrix}$$



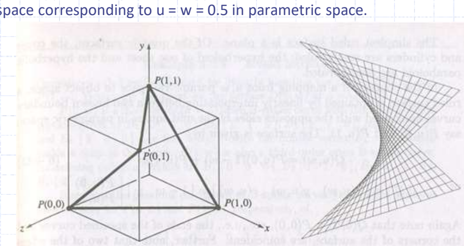


Bilinear surface

Determine the point on the bilinear surface defined by

P(0, 0)= [0 0 1],
P(0, 1)= [1 1 1],
P(1, 0)= [1 0 0],
P(1, 1)= [0 1 0],

i.e., the ends of opposite diagonals on opposite faces of a unit cube in object space corresponding to u = w = 0.5 in parametric space.



Bilinear surface

the surface in object space is a vector valued function

Q(u, w) = [x(u, w) y(u, w) z(u, w)]

then we have

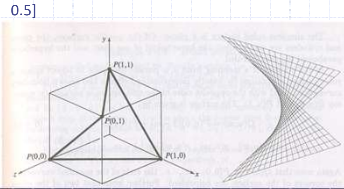
Q(0.5, 0.5) = [0 0 1](1-0.5)(1-0.5) + [1 1 1] (1-0.5) (0.5)

+ [1 0 0] (0.5) (1-0.5) + [0 1 0] (0.5) (0.5)

= 0.25 [0 0 1] + 0.25 [1 1 1]+ 0.25 [1 0 0]

+ 0.25 [0 1 0]

= [0.5 0.5 0.5]



Ruled and Developable Surfaces

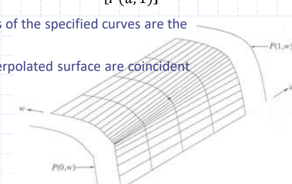
- A ruled surface is generated by a straight line moving along a path with one degree of freedom.
- Ruled surface is obtained by linearly interpolating between two known boundary curves associated with the opposite sides of the unit square in parametric space, say P(u, 0) and P(u, 1).

  The surface is given by

  Q(u, w) = P(u, 0) (1-w) + P(u, 1)w

  or $[Q] = [x(u,w) \quad y(u,w) \quad z(u,w)] = [1-w \quad w]\begin{bmatrix} P(u,0) \\ P(u,1) \end{bmatrix}$
- Again Q(0, 0)=P(0, 0), etc., i.e., the ends of the specified curves are the corners of the surface, are coincident.
- Two of the edges of the blended or interpolated surface are coincident with the given curves, i.e.,

  Q(u, 0) = P(u, 0) and Q(u, 1) = P(u, 1)



Ruled and Developable Surfaces

Alternately,

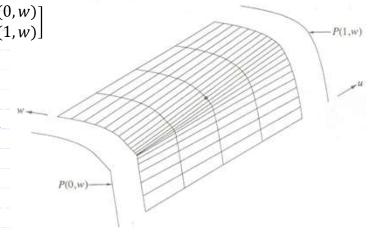
the curves corresponding to P(0, w) and P(1, w) are assumed known. The ruled surface is given by

Q(u, w) = P(0, w) (1-u) + P(1, w)u

or $[Q] = [x(u,w) \quad y(u,w) \quad z(u,w)]$

$$= [1-u \quad u]\begin{bmatrix} P(0,w) \\ P(1,w) \end{bmatrix}$$

Again the corners of the surface are coincident with the ends of the given curve, and the appropriate edges of the blended surface are coincident with the given boundary curves.



Ruled and Developable Surfaces

Consider a ruled surface formed by linearly blending the curves

P(0, w) and P(1, w).

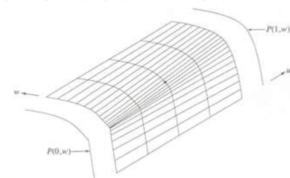
Determine the point on the surface Q(u, w) at u=w=0.5.

P(0, w) is a third-order (k=3) open B-spline curve with defining polygon vertices given by B_1[0 0 0], B_2[1 1 0], B_3[1 1 0], B_4[2 1 0] and B_5[3 0 0].

Notice the double vertex B2=B3 which yields a cusp in the curve.

P(1, w) is also a third order open B-spline curve.

Its defining polygon vertices are B'_1[0 0 6], B'_2[1 1 6], B'_3[2 1 6], B'_4[3 0 6].



Ruled and Developable Surfaces

B-spline curves yields open uniform knot vectors for P(0, w) and P(1, w), respectively, of

[X] = [0 0 0 1 2 3 3 3]

[Y] = [0 0 0 1 2 2 2]

the un-normalized parameter ranges for the two curves are different,

0 ≤ t ≤ 3 for P(0, w) and 0 ≤ s ≤ 2 for P(1, w).

The 'normalized' value for the ruled surface Q at w=0.5 corresponds to t=1.5 for P(0, w) and to s=1.0 for P(1, w).

P(0, w) yields:

P(0, w) = P(t)

$= B_1N_{1,3}(t) + B_2N_{2,3}(t) + B_3N_{3,3}(t) + B_4N_{4,3}(t) + B_5N_{5,3}(t)$

At w=0.5 or t=1.5

$P(0, 0.5) = P(1.5)= (0)B_1 + 0.125B_2 + 0.75B_3 + 0.125B_4 + (0)B_5$

= 0.125 [1 1 0] + 0.75 [1 1 0] + 0.125[2 1 0]

= [1.125 1 0]

Ruled and Developable Surfaces

Similarly,

P(1,w) = P(s)

$= B_1'N'_{1,3} + B'_2N'_{2,3} + B'_3N'_{3,3}(s) + B'_4N'_{4,3}(s)$

At w = 0.5 or s = 1.0

$P(1, 0.5) = P(1.0) = (0)B'_1 + 0.5B'_2 + 0.5B'_3 + (0)B'_4$

= 0.5[1 1 6] + 0.5[2 1 6]

= [1.5 1 6]

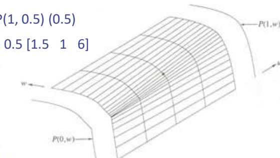
To obtain the point on the ruled surface yields

Q(u, w) = P(0, w) (1-u) + P(1, w) u

and Q(0.5, 0.5) = P(0, 0.5) (1-0.5) + P(1, 0.5) (0.5)

= 0.5 [1.125 1 0] + 0.5 [1.5 1 6]

= [1.3125 1 3]

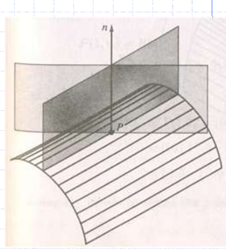


Ruled and Developable Surfaces

- A ruled surface is generated by a straight line moving along a path with one degree of freedom.
- Ruled surface is identified using the following technique.
- At any point on the surface, rotate a plane containing the normal to the surface at that point about the normal.

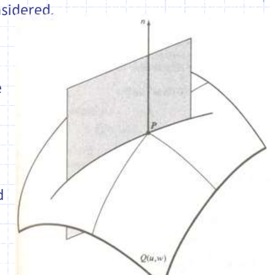
If there is at least one orientation in which every point on the edge of the plane contacts the surface, the surface is ruled in that direction.

If the edge of the rotating plane completely touches the surface in more than one orientation, the surface is multiply ruled at that point.



Developable Surfaces

- All ruled surfaces are not developable surface, however all developable surfaces are ruled.
- To find whether surface or a portion of a surface is developable or not curvature of parametric surface is considered.
- At any point P on a surface, the curve of intersection of a plane containing the normal to the surface at P and the surface, has a curvature K.
- As the plane is rotated about the normal, the curvature changes.
- There will be unique directions for which the curvature is a minimum and a maximum exists.



Developable Surfaces

- Euler the great Swiss mathematician, Showed that unique directions for which the curvature is a minimum and a maximum exists.
- The curvatures in these directions are called the principle curvatures, K_{min} and K_{max}.
- The principal curvature directions are orthogonal.
- Two combinations of the principal curvatures are of particular interest,
  - Average

$$H = \frac{\kappa_{\min} + \kappa_{\max}}{2}$$

  and

  - Gaussian curvatures.

$$K = \kappa_{\min} \ \kappa_{\max}$$

- *For a developable surface the Gaussian curvature K is everywhere zero, i.e., K=0.*

Developable Surfaces

- Dill has shown that for bi-parametric surfaces the average and Gaussian curvatures are given by:

$$H = \frac{A|Q_w|^2 - 2BQ_u \cdot Q_w + C|Q_u|^2}{2|Q_u \times Q_w|^3}$$

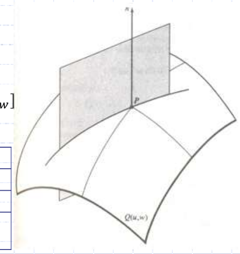
$$K = \frac{AC - B^2}{|Q_u \times Q_w|^4}$$

where

$$(A \ \ B \ \ C) = [Q_u \times Q_w] \cdot [Q_{uu} \quad Q_{uw} \quad Q_{ww}]$$

Effect of Gaussian curvature

$K_{min}K_{max}$	K	Shape
Same Sign	> 0	Elliptic (bump or hollow)
Opposite sign	< 0	Hyperbolic (saddle point)
One or both zero	0	Cylindrical/ conical (ridge, hollow, plane)



Show that an elliptic cone is a developable surface.

- Rewriting equation for a parametric elliptic cone in terms of u and w yields

$$Q(u,w) = [au\cos w \quad bu\sin w \quad cu]$$

- The partial derivatives are:

$$Q_u = [a\cos w \quad b\sin w \quad c]$$
$$Q_w = [-au\sin w \quad bu\cos w \quad 0]$$
$$Q_{uw} = [-a\sin w \quad b\cos w \quad 0]$$
$$Q_{uu} = [0 \quad 0 \quad 0]$$
$$Q_{ww} = [-au\cos w \quad -bu\sin w \quad 0]$$

$$Q_u \times Q_w = [-bcu\cos w \quad -acu\sin w \quad abu]$$

$$|Q_u \times Q_w|^2 = (abu)^2\left\{\left(\frac{c}{a}\cos w\right)^2 + \left(\frac{c}{b}\sin w\right)^2 + 1\right\} \neq 0 \quad u > 0$$

And

$$A = [-bcu\cos w \quad -acu\sin w \quad abu] \cdot [0 \quad 0 \quad 0] = 0$$

$$B = [-bcu\cos w \quad -acu\sin w \quad abu] \cdot [-a\sin w \quad b\cos w \quad 0]$$
$$= abcu\sin w\, cow - abcu\sin w\cos w = 0$$

$$C = [-bcu\cos w \quad -acu\sin w \quad abu] \cdot [-au\cos w \quad -bu\sin w \quad 0]$$
$$= abcu^2\cos^2 w + abcu^2\sin^2 w = abcu^2$$

Hence, using equation

$$K = \frac{AC - B^2}{|Q_u \times Q_w|^4}$$

$$= \frac{(0)(abcu^2) - (0)}{|Q_u \times Q_w|^4} = 0$$

everywhere on the surface,

Hence the surface is developable.

Linear Coons Surface

If the four boundary curves P(u, 0), P(u, 1), P(0, w) and P(1, w) are known and a bilinear blending function is used for the interior of the surface patch, a linear Coons surface is obtained.

At first glance it might be assumed that a simple sum of the singly ruled surface in the two directions u, w would yield the desired result. If this is done then

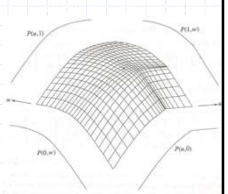
Q(u, w) = P(u, 0) (1-w) + P(u, 1)w + P(0, w) (1-u) + P(1, w)u

at the corners of the surface patch yields, e.g.,

Q(0, 0) = P(0, 0) + P(0, 0)=2P(0, 0)

and at the edges, e.g.,

Q(0, w) = P(0, 0) (1-w) + P(0, 1)w + P(0, w)



Linear Coons Surface

The correct result is obtained by subtracting the excess contribution to the surface due to duplication of the corner point.

This yield

Q(u, w) = P(u, 0) (1-w) + P(u, 1)w + P(0, w) (1-u) + P(1, w)u
– P(0, 0) (1-u) (1-w) – P(0, 1) (1-u)w – P(1, 0)u(1-w) – P(1, 1)uw

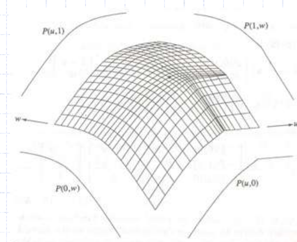
Here, at the corners,

Q(0, 0) = P(0, 0), etc.

and along the boundaries,

Q(0, w) = P(0, w)

Q(u, 1) = P(u, 1)



Linear Coons Surface

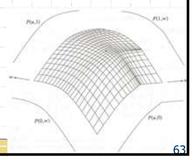
In matrix form, equation is

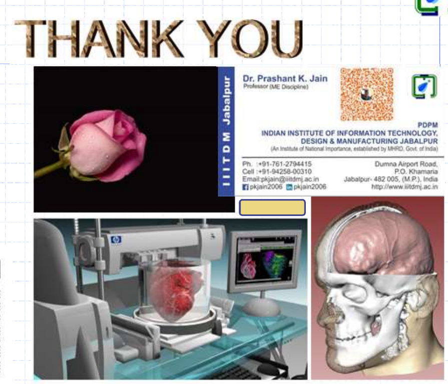
$$Q(u,w) = [1-u \quad u]\begin{bmatrix} P(0,w) \\ P(1,w) \end{bmatrix} + [P(u,0) \quad P(u,1)]\begin{bmatrix} 1-w \\ w \end{bmatrix}$$

$$-[1-u \quad u]\begin{bmatrix} P(0,0) & P(0,1) \\ P(1,0) & P(1,1) \end{bmatrix}\begin{bmatrix} 1-w \\ w \end{bmatrix}$$

or more compactly as

$$Q(u,w) = [1-u \quad u \quad 1]\begin{bmatrix} -P(0,0) & -P(0,1) & P(0,w) \\ -P(1,0) & -P(1,1) & P(1,w) \\ P(u,0) & P(u,1) & 0 \end{bmatrix}\begin{bmatrix} 1-w \\ w \\ 1 \end{bmatrix}$$



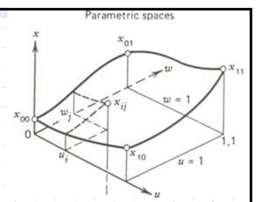


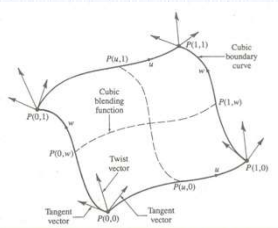
Coons Bi-cubic Surface

- **Linear Coons Surface**
  - Are not sufficiently flexible for many applications.
- The coons bi-cubic surface patch uses normalized cubic splines for all four boundary curves.
- Cubic blending functions are used to define the interior of the patch.
- Thus, each boundary curve is of the general form

$P(t) = B_1 + B_2t + B_3t^2 + B_4t^3$

$0 \leq t \leq 1$





Coons Bi-cubic Surface

- For a single normalized cubic spline segment with known tangent and position vectors at the ends, each of the four boundary curves, P(u, 0), P(u, 1), P(0, w) and P(1, w), is given by

P(t) = [T] [N] [G]

$$= [t^3 \quad t^2 \quad t \quad 1]\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_1' \\ P_2' \end{bmatrix} \qquad 0 \leq t \leq 1$$

where t becomes u or w as appropriate and P_1, P_2, P_1', P_2' are the position and tangent vectors at the ends of the appropriate boundary curve.

Coons Bi-cubic Surface

- The cubic blending function used for both parametric directions is identical to the one used to blend the interior of a normalized cubic spline curve; i.e.,

$[F] = [\ F_1(t) \quad F_2(t) \quad F_3(t) \quad F_4(t)\ ] = [T]\ [N]$

$$= [t^3 \quad t^2 \quad t \quad 1]\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Blending functions are as:

$F_1(t) = 2t^3 - 3t^2 + 1$

$F_2(t) = -2t^3 + 3t^2$

$F_3(t) = t^3 - 2t^2 + t$

$F_4(t) = t^3 - t^2$

Where t is either u or w as appropriate.

Coons Bi-cubic Surface

- The definition for a Coons bi-cubic patch is

$$Q(u,w) = [F_1(u) \quad F_2(u) \quad F_3(u) \quad F_4(u)] \times \begin{bmatrix} P(0,0) & P(0,1) & P_w(0,0) & P_w(0,1) \\ P(1,0) & P(1,1) & P_w(1,0) & P_w(1,1) \\ P_u(0,0) & P_u(0,1) & P_{uw}(0,0) & P_{uw}(0,1) \\ P_u(1,0) & P_u(1,1) & P_{uw}(1,0) & P_{uw}(1,1) \end{bmatrix}\begin{bmatrix} F_1(w) \\ F_2(w) \\ F_3(w) \\ F_4(w) \end{bmatrix}$$

for $0 \leq u \leq 1$ and $0 \leq w \leq 1$.

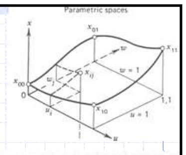
- The definition can be more compactly written as

$Q(u, w) = [U]\ [N]\ [P]\ [N]^T\ [W]$

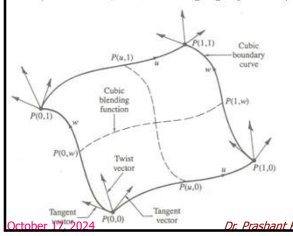
- where $[U] = [u^3 \quad u^2 \quad u \quad 1]$ and $[W]^T = [w^3 \quad w^2 \quad w \quad 1]$

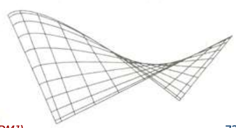
Coons Bi-cubic Surface

- A bi-cubic surface patch is defined by the four position vectors at the corners, eight tangent vectors, two at each corner, the four twist vectors at the corners, and the four cubic blending functions F_1, F_2, F_3 and F_4.



$$[P] = \left[\begin{array}{c:c} \text{corner position vectors} & w-\text{tangent vectors} \\ \hdashline u-\text{tangent vectors} & \text{twist vectors} \end{array}\right]$$





Derivatives of Bi-cubic Surface

The parametric derivatives at any point on a bi-cubic surface are obtained by formally differentiating

$Q_u(u, w) = [U'] [N] [P] [N]^T [W]$

$Q_w(u, w) = [U] [N] [P] [N]^T [W']$

$Q_{uw}(u, w) = [U'] [N] [P] [N]^T [W']$

$Q_{uu}(u, w) = [U''] [N] [P] [N]^T [W]$

$Q_{ww}(u, w) = [U] [N] [P] [N]^T [W'']$

Where the primes denote differentiation with respect to the appropriate variable and

$[U'] = [3u^2 \quad 2u \quad 1 \quad 0]$

$[W']^T = [3w^2 \quad 2w \quad 1 \quad 0]$

$[U''] = [6u \quad 2 \quad 0 \quad 0]$

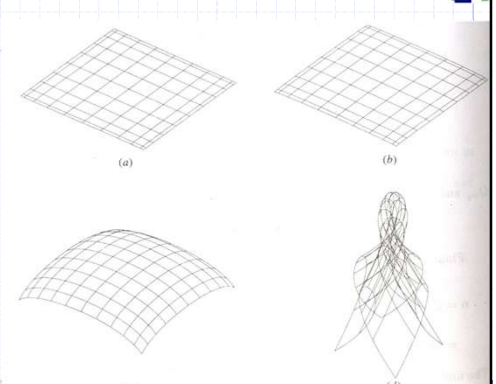
$[W'']^T = [6w \quad 2 \quad 0 \quad 0]$

The normal to the surface, which is important in hidden surface, illumination model, and numerical control calculations, is given by

$n = Q_u \times Q_w$

Effect of tangent vector magnitude

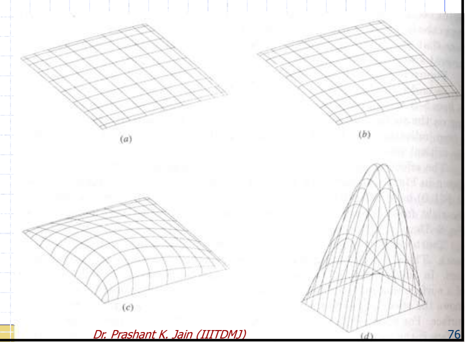
(a) 1;
(b) 10;
(c) 100;
(d) 1000



Effect of twist vector magnitude

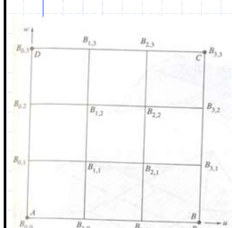
The magnitude of the twist vector component at P(0, 0) is

(a) 10;
(b) 100;
(c) 1000;
(d) 10,000.

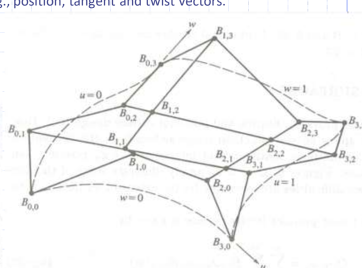


Bezier Surfaces

Coons bicubic surfaces provide a flexible and powerful surface design tool. However, practical usage suffers from the necessity of specifying precise, non-intuitive mathematical information, e.g., position, tangent and twist vectors.



Schematic of defining polygon net



Bezier surface nomenclature

Bezier Surfaces

A Bezier surface is given by

$$Q(u, w) = \sum_{i=0}^{n} \sum_{j=0}^{m} B_{i,j} J_{n,i}(u) K_{m,j}(w)$$

where $J_{n,i}(u)$ and $K_{m,j}(w)$ are the Bernstein basis functions in the u and w parametric directions. Repeating the definition previously given for convenience yields

$$J_{n,i}(u) = \binom{n}{i} u^i (1-u)^{n-i} \quad \text{and} \quad K_{m,j}(w) = \binom{m}{j} w^j (1-w)^{m-j}$$

with

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \quad \text{and} \quad \binom{m}{j} = \frac{m!}{j!(m-j)!}$$

The $B_{i,j}$'s are the vertices of a defining polygon net in u and w directions, respectively.

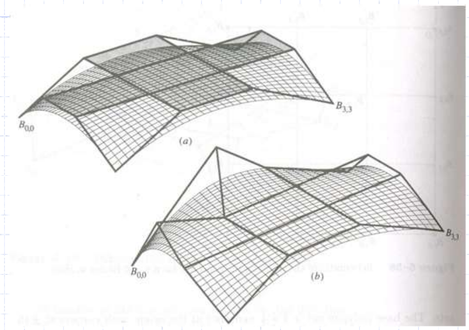
For quadrilateral surface patches the defining polygon net must be topologically rectangular, i.e., the net must have the same number of vertices in each 'row'.

Bezier Surfaces

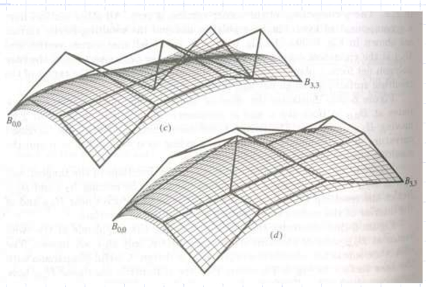
Many properties of the surface are similar to curves and are known.

- The degree of the surface in each parametric direction is one less than the number of defining polygon vertices in that direction.
- The continuity of the surface in each parametric direction is two less than the number of defining polygon vertices in that direction.
- The surface generally follows the shape of the defining polygon net.
- Only the corner points of the defining polygon net and the surface are coincident.
- The surface is contained within the convex hull of the defining polygon net.
- The surface does not exhibit the variation diminishing property. The variation diminishing property for bi-variant surfaces is both undefined and unknown.
- The surface is invariant under an affine transformation.

Effect of increasing magnitude of tangent vector



Effect of change in tangent vector direction and magnitude of twist vector



Bezier Surfaces

In matrix form a Cartesian product Bezier surface is given by

Q(u, w) = [U] [N] [B] [M]^T [W]

where

$$[U]=[u^n \quad u^{n-1} \quad \dots \quad 1]$$

$$[W]=[w^m \quad w^{m-1} \quad \dots \quad 1]^T$$

$$[B]=\begin{bmatrix} B_{0,0} & \dots & B_{0,m} \\ \dots & \dots & \dots \\ B_{n,0} & \dots & B_{n,m} \end{bmatrix}$$

Bezier Surfaces

For the specific case of a 4x4 bicubic Bezier surface

Equation reduces to

$$Q(u,w)=[u^3 \quad u^2 \quad u \quad 1]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} B_{0,0} & B_{0,1} & B_{0,2} & B_{0,3} \\ B_{1,0} & B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,0} & B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,0} & B_{3,1} & B_{3,2} & B_{3,3} \end{bmatrix} \times$$

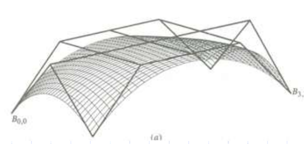
$$\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} w^3 \\ w^2 \\ w \\ 1 \end{bmatrix}$$

Bezier Surfaces

- A Bezier surface need not be square.
- For a 5x3 net equation yields

$$Q(u,w)=[u^4 \quad u^3 \quad u^2 \quad u \quad 1]\begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & -12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \times$$

$$\begin{bmatrix} B_{0,0} & B_{0,1} & B_{0,2} \\ B_{1,0} & B_{1,1} & B_{1,2} \\ B_{2,0} & B_{2,1} & B_{2,2} \\ B_{3,0} & B_{3,1} & B_{3,2} \\ B_{4,0} & B_{4,1} & B_{4,2} \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} w^2 \\ w \\ 1 \end{bmatrix}$$



Derivatives of a Bezier surface

- The derivatives of a Bezier surface are obtained by formal differentiation of equations.
- First and second parametric derivatives are

$$Q_u(u,w)=\sum_{i=0}^{n}\sum_{j=0}^{m}B_{i,j}J'_{n,i}(u)K_{m,j}(w)$$

$$Q_w(u,w)=\sum_{i=0}^{n}\sum_{j=0}^{m}B_{i,j}J_{n,i}(u)K'_{m,j}(w)$$

$$Q_{uw}(u,w)=\sum_{i=0}^{n}\sum_{j=0}^{m}B_{i,j}J'_{n,i}(u)K'_{m,j}(w)$$

$$Q_{uu}(u,w)=\sum_{i=0}^{n}\sum_{j=0}^{m}B_{i,j}J''_{n,i}(u)K_{m,j}(w)$$

$$Q_{ww}(u,w)=\sum_{i=0}^{n}\sum_{j=0}^{m}B_{i,j}J_{n,i}(u)K''_{m,j}(w)$$

Bezier surface

The relationship between a bi-cubic Bezier and bi-cubic Coons surface is easily found.

Recalling equations and equating them yields

$Q_{Coons}(u, w) = Q_{Bezier}(u, w)$

$[U]\,[N_C]\,[P]\,[N_C]^T\,[W] = [U]\,[N_B]\,[B]\,[N_B]^T\,[W]$

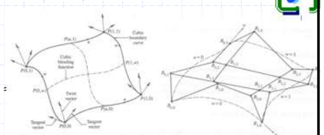
where $[N_C]$ and $[N_B]$ are known equations. Hence the bicubic Coons surface geometric matrix [P] is given in terms of the Bezier surface polygon net as

$$[P]=[N_C]^{-1}[N_B][B][N_B]^T\left[[N_C]^T\right]^{-1}$$

Bezier surface

In matrix form

$$\begin{bmatrix} P(0,0) & P(0,1) & P_w(0,0) & P_w(0,1)\\ P(1,0) & P(1,1) & P_w(1,0) & P_w(1,1)\\ P_u(0,0) & P_u(0,1) & P_{uw}(0,0) & P_{uw}(0,1)\\ P_u(1,0) & P_u(1,1) & P_{uw}(0,1) & P_{uw}(1,1)\end{bmatrix}=$$

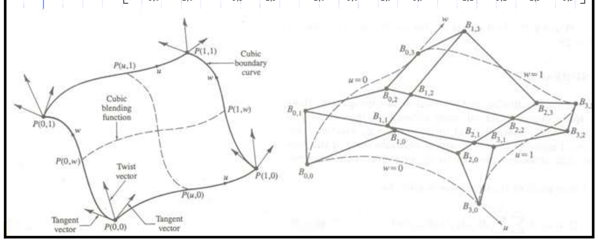


$$=\begin{bmatrix} B_{0,0} & B_{0,3} & 3(B_{0,1}-B_{0,0}) & 3(B_{0,3}-B_{0,2})\\ B_{3,0} & B_{3,3} & 3(B_{3,1}-B_{3,0}) & 3(B_{3,3}-B_{3,2})\\ 3(B_{1,0}-B_{0,0}) & 3(B_{1,3}-B_{0,3}) & 9(B_{0,0}-B_{1,0}-B_{0,1}+B_{1,1}) & 9(B_{0,2}-B_{1,2}-B_{0,3}+B_{1,3})\\ 3(B_{3,0}-B_{2,0}) & 3(B_{3,3}-B_{2,3}) & 9(B_{2,0}-B_{3,0}-B_{2,1}+B_{3,1}) & 9(B_{2,2}-B_{3,2}-B_{2,3}+B_{3,3})\end{bmatrix}$$

- Lower right 2x2 submatrix confirms that the center four defining polygon net vertices influence the twist at the bi-cubic Bezier patch corners.
- The twist at a corner controlled by not only the center polygon vertices but also by the adjacent tangent vectors.
- In fact, the twist at the corner is controlled by the shape of the nonplanar quadrilateral formed by the corner, the two adjacent boundary points and the adjacent center point.

$$\begin{bmatrix} P(0,0) & P(0,1) & P_w(0,0) & P_w(0,1)\\ P(1,0) & P(1,1) & P_w(1,0) & P_w(1,1)\\ P_u(0,0) & P_u(0,1) & P_{uw}(0,0) & P_{uw}(0,1)\\ P_u(1,0) & P_u(1,1) & P_{uw}(0,1) & P_{uw}(1,1)\end{bmatrix}=$$

$$=\begin{bmatrix} B_{0,0} & B_{0,3} & 3(B_{0,1}-B_{0,0}) & 3(B_{0,3}-B_{0,2})\\ B_{3,0} & B_{3,3} & 3(B_{3,1}-B_{3,0}) & 3(B_{3,3}-B_{3,2})\\ 3(B_{1,0}-B_{0,0}) & 3(B_{1,3}-B_{0,3}) & 9(B_{0,0}-B_{1,0}-B_{0,1}+B_{1,1}) & 9(B_{0,2}-B_{1,2}-B_{0,3}+B_{1,3})\\ 3(B_{3,0}-B_{2,0}) & 3(B_{3,3}-B_{2,3}) & 9(B_{2,0}-B_{3,0}-B_{2,1}+B_{3,1}) & 9(B_{2,2}-B_{3,2}-B_{2,3}+B_{3,3})\end{bmatrix}$$



Bezier surface

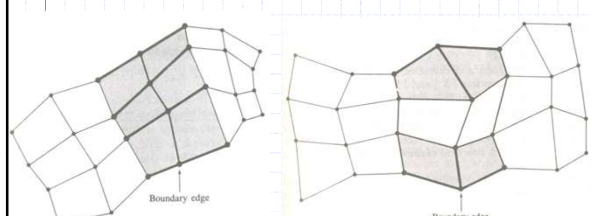
- Similarly the inverse relationship between [P] and [B] which gives the Bezier polygon net vertices in terms of the Coons bi-cubic surface parameters is

$$\begin{bmatrix} B_{0,0} & B_{0,1} & B_{0,2} & B_{0,3}\\ B_{1,0} & B_{1,1} & B_{1,2} & B_{1,3}\\ B_{2,0} & B_{2,1} & B_{2,2} & B_{2,3}\\ B_{3,0} & B_{3,1} & B_{3,2} & B_{3,3}\end{bmatrix}=$$

$$\left(\frac{1}{3}\right)\begin{bmatrix} 3P(0,0) & 3P(0,0)+P_w(0,0) & 3P(0,1)-P_w(0,1) & 3P(0,1)\\ 3P(0,0)+P_u(0,0) & \frac{1}{3}\{P_{uw}(0,0)+9P(0,0)-3P_w(0,0)+P_u(0,0)\} & \frac{1}{3}\{P_{uw}(0,1)+9P(0,1)+3(P_u(0,1)-P_w(0,1))\} & 3P(0,1)+P_u(0,1)\\ 3P(1,0)+P_u(1,0) & \frac{1}{3}\{P_{uw}(1,0)+9P(1,0)+3P_w(1,0)-P_u(1,0)\} & \frac{1}{3}\{P_{uw}(1,1)+9P(1,1)-3(P_u(1,1)+P_w(1,1))\} & 3P(1,1)-P_u(1,1)\\ 3P(1,0) & 3P(1,0)+P_w(1,0) & 3P(1,1)-P_w(1,1) & 3P(1,1)\end{bmatrix}$$

Joining two Bezier Patch

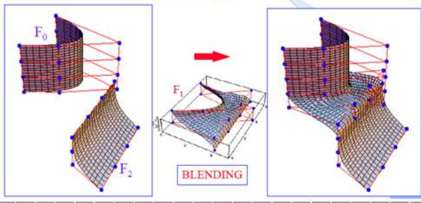
- For C^0 continuity along the edge of two boundary curves and hence two boundary polygons along edge must be coincident.
- For C^1 continuity across the patch boundary, surface normal direction along boundary edge must be same. Two conditions may be used to achieve this:
  - Four polygon net lines that meet at and cross the boundary edge to be collinear.
  - Three polygon net edges meeting at the ends of the boundary curve to be coplanar.

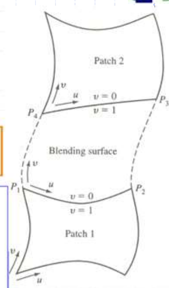


Blending of two Bezier surface patches

If a single surface does not approximate enough a given set of points, we may use several patches joined together.

Two Bézier patches F_0 and F_2 are connected with C^1-continuity by using a Bézier F_1 patch.





B Spline Surface

$$Q(u,w)=\sum_{i=1}^{n+1}\sum_{j=1}^{m+1}B_{i,j}N_{i,k}(u)M_{j,l}(w)$$

where, $N_{i,k}(u)$ and $N_{j,l}(w)$ are the B-spline basis functions in the bi-parametric u and w directions, respectively.

The definition for the basis functions given as

$$N_{i,1}(u)=\begin{cases}1 & if\ x_i \le u \le x_{i+1}\\ 0 & otherwise\end{cases}$$

$$N_{i,k}(u)=\frac{(u-x_i)N_{i,k-1}(u)}{x_{i+k-1}-x_i}+\frac{(x_{i+k}-u)N_{i+1,k-1}(u)}{x_{i+k}-x_{i+1}}$$

$$M_{j,1}(w)=\begin{cases}1 & if\ y_j \le w \le y_{j+1}\\ 0 & otherwise\end{cases}$$

$$N_{j,l}(w)=\frac{(w-y_j)m_{j,l-1}(w)}{y_{j+l-1}-y_j}+\frac{(y_{j+l}-w)M_{j+1,l-1}(w)}{y_{j+l}-y_{j+1}}$$

where the x_i and y_j are elements of knot vectors.

Again the $B_{i,j}$'s are the vertices of a defining polygon net.

Properties of B Spline Surface

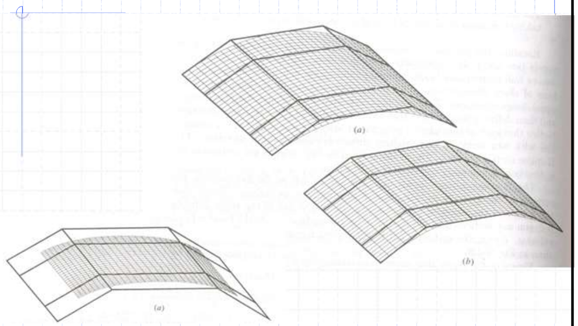
- Non-uniform knot vectors can be used in both parametric directions.
- Although it is common to use the same type of knot vectors in both parametric directions, it is not required.
- Because the B-spline basis is used to describe both the boundary curves and to blend the interior of the surface, several properties of the B-spline surface are immediately known:

Properties of B Spline Surface

- The maximum order of the surface in each parametric direction is equal to the number of defining polygon vertices in that direction.
- The continuity of the surface in each parametric direction is two less than the order in each direction; i.e., C^{k-2} and C^{l-2} in the u and w directions, respectively.
- The surface is invariant with respect to an affine transformation; i.e., the surface is transformed by transforming the defining polygon net.
- The variation diminishing property for B-spline surfaces is currently not known.
- The influence of a single polygon net vertex is limited to $\pm k/2, \pm l/2$ spans in each parametric direction.
- If the number of defining polygon net vertices is equal to the order in each parametric direction and there are no interior knot values, then the B-spline surface reduces to a Bezier surface.
- If triangulated, the defining polygon net forms a planar approximation to the surface.
- The surface lies within the convex hull of the defining polygon net formed by taking the union of all convex hulls of K, l neighbouring polygon net veritces.

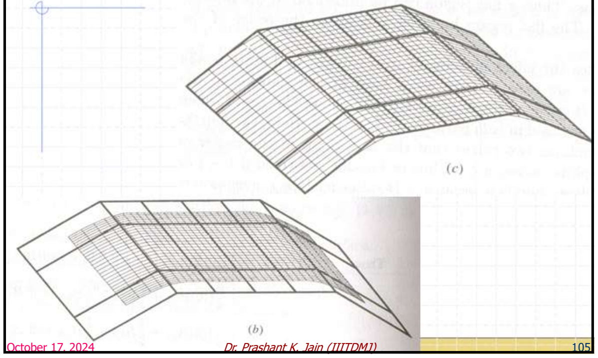
Effect of collinear vertices

- Small interior flat region due to three collinear net vertices in u



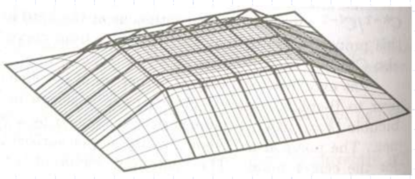
Effect of collinear vertices

- Larger interior flat region due to five collinear net vertices in u



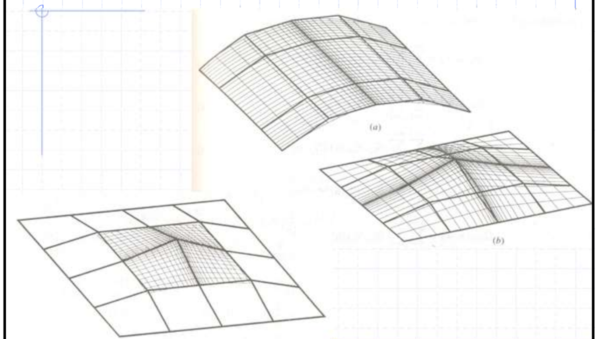
Effect of collinear vertices

- Flat region embedded within sculptured surface



Effect of multiple coincident net lines

- Fourth order B-Spline surfaces with multiple coincident net lines



Local control in B-Spline surfaces

The excellent local control properties of B-spline curves carry over to B-spline surfaces.

an open bicubic (k=l=4) B-spline surface is defined by a 9x9 (m=n=8) polygon net. The polygon net, is flat except for the center point.

The open knot vector in both parametric directions is

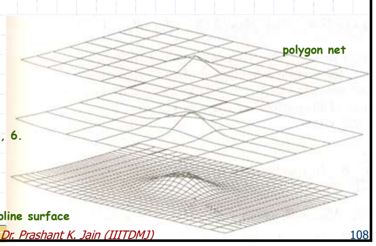
[0 0 0 0 1 2 3 4 5 6 6 6 6].

Thus, there are six parametric spans in each direction, i.e.,

0 - 1, 1 - 2, ..., 5 - 6.

Each parametric quadrilateral, e.g., $0 \le u \le 1$, $0 \le w \le 1$, forms a B-spline surface subpatch.

The influence of the displaced point is confined to $\pm k/2, l/2$ spans or sub-patches.



Properties of B Spline Surface

- The parametric derivatives of a B-spline surface are obtained by formally differentiating.

$$Q_u(u,w) = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} B_{i,j} N'_{i,k}(u) M_{j,l}(w)$$

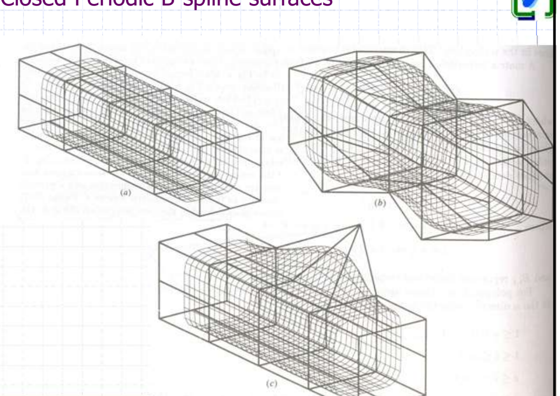
$$Q_w(u,w) = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} B_{i,j} N'_{i,k}(u) M_{j,l}(w)$$

$$Q_{uw}(u,w) = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} B_{i,j} N'_{i,k}(u) M'_{j,l}(w)$$

$$Q_{uu}(u,w) = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} B_{i,j} N''_{i,k}(u) M_{j,l}(w)$$

$$Q_{ww}(u,w) = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} B_{i,j} N_{i,k}(u) M''_{j,l}(w)$$

Closed Periodic B-spline surfaces



Rational B-spline surfaces

$$Q(u,w) = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} B^h{}_{i,j} N_{i,k}(u) M_{j,l}(w)$$

where the $B^h{}_{i,j}$'s are the 4D homogenous defining polygon vertices and $N_{i,k}(u)$ and $M_{j,l}(w)$ are the non-rational B-spline basis functions.

- Projecting back into three-dimensional space by dividing through by the homogenous coordinate gives the rational B-spline surface.

$$Q(u,w) = \frac{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1} h_{i,j} B^h{}_{i,j} N_{i,k}(u) M_{j,l}(w)}{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1} h_{i,j} N_{i,k}(u) M_{j,l}(w)} = Q(u,w) = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} B_{i,j} S_{i,j}(u,w)$$

where the $B_{i,j}$'s are the 3D defining polygon net points and the $S_{i,j}(u, w)$ are bivariant rational B-spline surface basis functions

$$S_{i,j}(u,w) = \frac{h_{i,j} N_{i,k}(u) M_{j,l}(w)}{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1} h_{i1,j1} N_{i1,k}(u) M_{j1,l}(w)}$$

It is convenient to assume $h_{i,j} \ge 0$ for all i, j.

Properties of Rational B-spline surfaces

Spline surfaces have similar analytic and geometric properties to their nonrational counterparts.

Specifically,

- The sum of the rational surface basis functions for any u, w values is

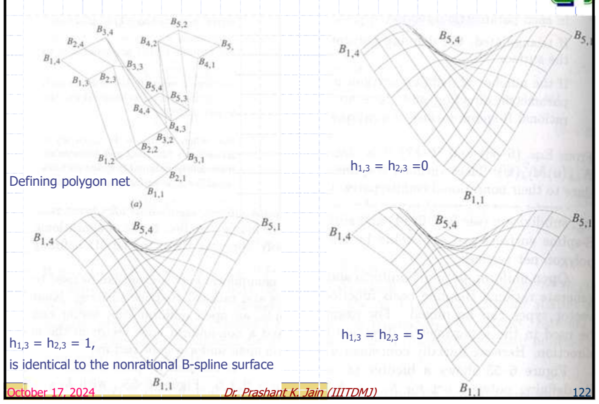
$$\sum_{i=1}^{n+1}\sum_{j+1}^{m+1} S_{i,j}(u,w) \equiv 1$$

- Each rational surface basis function is positive or zero for all parameter values u, w, i.e., $S_{i,j} \geq 0$.
- Except for k=1 or l=1, each rational surface basis function has precisely one maximum.
- The maximum order of a rational B-spline surface in each parametric direction is equal to the number of defining polygon vertices in that direction.
- A rational B-spline surface of order k, l (degree k-1, l-1) is C^{k-2} , C^{l-2} continuous everywhere.

Properties of Rational B-spline surfaces

- A rational B-spline surface is invariant with respect to a projective transformation; i.e., any projective transformation can be applied to the surface by applying it to the defining polygon net. Note this is a stronger condition than that for a nonrational B-spline surface.
- The surface lies within the convex hull of the defining polygon net formed by taking the union of all convex hull of k, l neighboring polygon net vertices.
- The variation diminishing property is not known for rational B-spline surface.
- The influence of a single polygon net vertex is limited to ±k/2, ±l/2 spans in each parametric direction.
- If triangulated, the defining polygon net forms a planar approximation to the surface.
- If the number of defining polygon net vertices is equal to the order in each parametric direction and there are no duplicate interior knot values, the rational B-spline surface is a rational Bezier surface.

Rational B-spline surfaces with n+1 = 5, m+1 = 4, k = l= 4



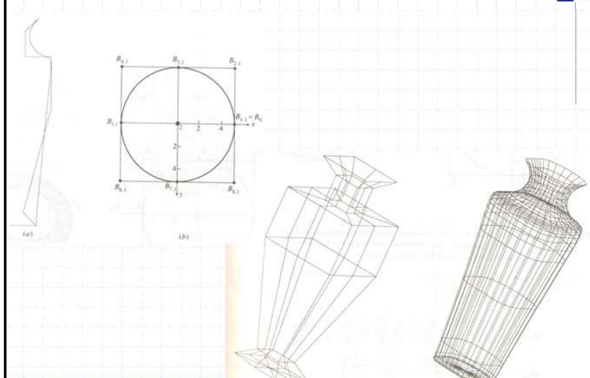
Defining polygon net

$h_{1,3} = h_{2,3} = 0$

$h_{1,3} = h_{2,3} = 1$, is identical to the nonrational B-spline surface

$h_{1,3} = h_{2,3} = 5$

Rational B-splined surface of revolution



Derivatives of a rational B-spline surface

- The derivatives of a rational B-spline surface are obtained by formal differentiation and the results are:

$$Q_u = \frac{\overline{N}}{\overline{D}}\left(\frac{\overline{N_u}}{\overline{N}} - \frac{\overline{D_u}}{\overline{D}}\right)$$

$$Q_w = \frac{\overline{N}}{\overline{D}}\left(\frac{\overline{N_w}}{\overline{N}} - \frac{\overline{D_w}}{\overline{D}}\right)$$

$$Q_{uw} = \frac{\overline{N}}{\overline{D}}\left(\frac{\overline{N_{uw}}}{\overline{N}} - \frac{\overline{N_u}}{\overline{N}}\frac{\overline{D_w}}{\overline{D}} - \frac{\overline{N_w}}{\overline{N}}\frac{\overline{D_u}}{\overline{D}} + 2\frac{\overline{D_u}}{\overline{D}}\frac{\overline{D_w}}{\overline{D}} - \frac{\overline{D_{uw}}}{\overline{D}}\right)$$

$$Q_{uu} = \frac{\overline{N}}{\overline{D}}\left(\frac{\overline{N_{uu}}}{\overline{N}} - 2\frac{\overline{N_u}}{\overline{N}}\frac{\overline{D_u}}{\overline{D}} + 2\frac{\overline{D_u}^2}{\overline{D}^2} - \frac{\overline{D_{uu}}}{\overline{D}}\right)$$

$$Q_{ww} = \frac{\overline{N}}{\overline{D}}\left(\frac{\overline{N_{ww}}}{\overline{N}} - 2\frac{\overline{N_w}}{\overline{N}}\frac{\overline{D_w}}{\overline{D}} + 2\frac{\overline{D_w}^2}{\overline{D}^2} - \frac{\overline{D_{ww}}}{\overline{D}}\right)$$

- Where $\overline{N}$ and $\overline{D}$ are the numerator and denominator, respectively, with derivatives

$$\overline{N} = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} h_{i,j}B_{i,j}N_{i,k}(u)M_{j,l}(w) \qquad \overline{D} = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} h_{i,j}N_{i,k}(u)M_{j,l}(w)$$

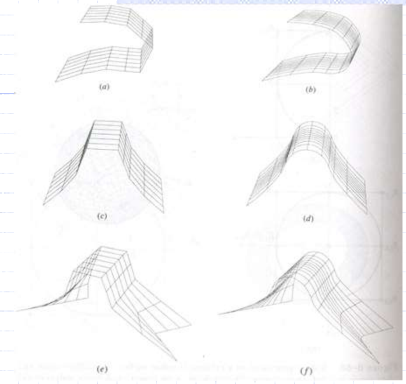
$$\overline{N_u} = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} h_{i,j}B_{i,j}N'_{i,k}(u)M_{j,l}(w) \qquad \overline{D_u} = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} h_{i,j}N'_{i,k}(u)M_{j,l}(w)$$

$$\overline{N_w} = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} h_{i,j}B_{i,j}N_{i,k}(u)M'_{j,l}(w) \qquad \overline{D_w} = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} h_{i,j}N_{i,k}(u)M'_{j,l}(w)$$

$$\overline{N_{uw}} = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} h_{i,j}B_{i,j}N'_{i,k}(u)M'_{j,l}(w) \qquad \overline{D_{uw}} = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} h_{i,j}N'_{i,k}(u)M'_{j,l}(w)$$

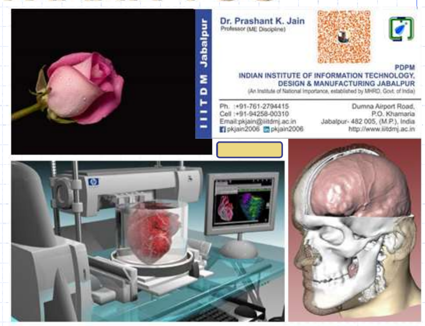
$$\overline{N_{uu}} = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} h_{i,j}B_{i,j}N''_{i,k}(u)M_{j,l}(w) \qquad \overline{D_{uu}} = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} h_{i,j}N''_{i,k}(u)M_{j,l}(w)$$

$$\overline{N_{ww}} = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} h_{i,j}B_{i,j}N_{i,k}(u)M''_{j,l}(w) \qquad \overline{D_{ww}} = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} h_{i,j}N_{i,k}(u)M''_{j,l}(w)$$



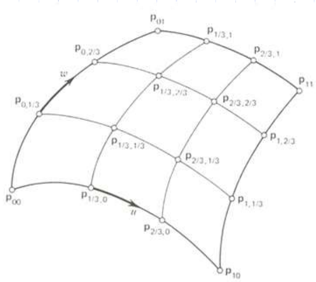
A quadratic surface element within a more general rational B-spline surface. Airfoil leading edge. (a) Polygon net; (b) surface. Ship stern (c) Polygon net; (d) surface. A cylinder as part of a more general surface. (e) Polygon net; (f) surface.

THANK YOU



Sixteen point form

- It may be difficult or impractical to provide tangents and twist vectors to define cubic patch
- 48 degrees of freedom or algebraic coefficients need to be supplied



Now expand the algebraic form of the patch $\mathbf{p} = \mathbf{UAW}^T$ to obtain

$$\mathbf{a}_{33}u^3w^3 + \mathbf{a}_{32}u^3w^2 + \ldots + \mathbf{a}_{00} = \mathbf{p}(u, w)$$

we can generate 16 of these equations, one for each of the 16 points.

Let us use the u, w values. Thus,

$$\mathbf{a}_{33}(0)^3(0)^3 + \mathbf{a}_{32}(0)^3(0)^2 + \ldots + \mathbf{a}_{00} = \mathbf{p}(0, 0)$$

$$\mathbf{a}_{33}(1/3)^3(0)^3 + \mathbf{a}_{32}(1/3)^3(0)^2 + \ldots + \mathbf{a}_{00} = \mathbf{p}(1/3, 0)$$

- -

- -

$$\mathbf{a}_{33}(2/3)^3(1/3)^3 + \mathbf{a}_{32}(2/3)^3(1/3)^2 + \ldots + \mathbf{a}_{00} = \mathbf{p}(2/3, 1/3)$$

- -

- -

$$\mathbf{a}_{33}(1)^3(1)^3 + \mathbf{a}_{32}(1)^3(1)^2 + \ldots + \mathbf{a}_{00} = \mathbf{p}(1, 1)$$

In matrix form, this set of equations becomes

$$Ea = p$$

$$a = E^{-1} p$$

Investigate the 16- point solution for the geometric form.

We can rewrite $p = UMBM^TW^T$ as $p = UNPN^TW^T$

The **B** matrix is replaced by a Matrix **P**

$$P = \begin{bmatrix} p(0,0) & p(0,1/3) & p(0,2/3) & p(0,1) \\ p(1/3,0) & p(1/3,1/3) & p(1/3,2/3) & p(1/3,1) \\ p(2/3),0 & p(2/3,1/3) & p(2/3,2/3) & p(2/3,1) \\ p(1,0) & p(1,1/3) & p(1,2/3) & p(1,1) \end{bmatrix}$$

By doing the indicated algebra, we find N to be

$$N = \begin{bmatrix} -9/2 & 27/2 & -27/2 & 9/2 \\ 9 & -45/2 & 18 & -9/2 \\ -11/2 & 9 & -9/2 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Same as found for B-spline curves

Degenerate Surfaces and Pathological Conditions

- Degenerate boundary curves
- Superimposed corners
- Internal degeneracies
- The simplest degenerate patch is a point where

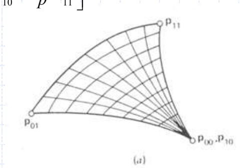
  $p_{00} = p_{10} = p_{01} = p_{11}$ and all other coefficients are identically equal to zero.
- A patch degenerates to a straight line when

  $p^u_{00} = p^u_{10} = p_{10} - p_{00}$ and all other coefficients are zero.

Degenerate Surfaces

For the three-sided patch if we assume that $p_{00} = p_{10}$, then the **B** matrix is

$$B_{(a)} = \begin{bmatrix} p_{00} & p_{01} & p^w{}_{00} & p^w{}_{01} \\ p_{00} & p_{11} & p^w{}_{10} & p^w{}_{11} \\ 0 & p^u{}_{01} & p^{uw}{}_{00} & p^{uw}{}_{01} \\ 0 & p^u{}_{11} & p^{uw}{}_{10} & p^{uw}{}_{11} \end{bmatrix}$$

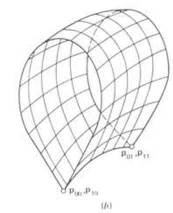


Degenerate Surfaces

Instead of a zero length fourth side, one of the sides does double duty.

Its B matrix might be as follows:

$$B_{(b)} = \begin{bmatrix} p_{00} & p_{01} & p^w{}_{00} & p^w{}_{01} \\ p_{00} & p_{01} & p^w{}_{00} & p^w{}_{01} \\ p^u{}_{00} & p^u{}_{01} & p^{uw}{}_{00} & p^{uw}{}_{01} \\ p^u{}_{10} & p^u{}_{11} & p^{uw}{}_{10} & p^{uw}{}_{11} \end{bmatrix}$$

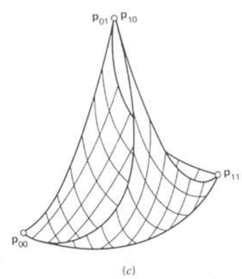


Degenerate Surfaces

When points diagonally opposite each other are coincident,

The B matrix would be as follows:

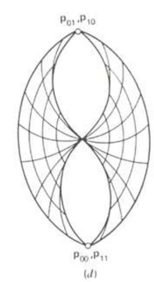
$$B_{(c)} = \begin{bmatrix} p_{00} & p_{10} & p^w{}_{00} & p^w{}_{01} \\ p_{10} & p_{11} & p^w{}_{10} & p^w{}_{11} \\ p^u{}_{00} & p^u{}_{01} & p^{uw}{}_{00} & p^{uw}{}_{01} \\ p^u{}_{10} & p^u{}_{11} & p^{uw}{}_{10} & p^{uw}{}_{11} \end{bmatrix}$$



Degenerate Surfaces

- An interesting shape may be obtained by joining the remaining diagonally opposite pair of points with the following **B** matrix:

$$B_{(d)} = \begin{bmatrix} p_{00} & p_{10} & p^w{}_{00} & p^w{}_{01} \\ p_{10} & p_{00} & p^w{}_{10} & p^w{}_{11} \\ p^u{}_{00} & p^u{}_{01} & p^{uw}{}_{00} & p^{uw}{}_{01} \\ p^u{}_{10} & p^u{}_{11} & p^{uw}{}_{10} & p^{uw}{}_{11} \end{bmatrix}$$



Curves on Surfaces

- A surface provides a two-dimensional space suitable for supporting the vector analytic representation of curves.
- Consider the curve on the surface. This curve is defined in the u, w parametric space of the patch.
- The points on the curve have vector components in the u and w directions. Thus,

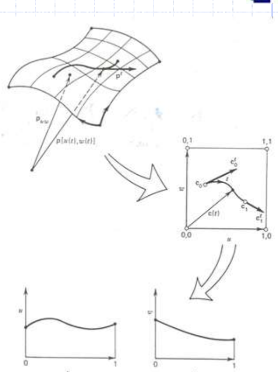
  C(t) = u(t) + w(t)

  Then, we define c(t) as

  $c(t) = TMB_c$

  $T = [t^3 \quad t^2 \quad t \quad 1]$

  $B_c = [c_0 \quad c_1 \quad c^t{}_0 \quad c^t{}_1]^T$



- To determine the coordinates of points on this curve in object space, first compute pairs of u, w values for successive values of t.
- Substitute these u, w values into the surface equation, say $p(u, w) = UMB_sM^TW^T$, to obtain a sequence of x, y, z coordinates of points in object space.
- These points are necessarily on the patch. This process selects only those points on the patch that satisfy the curve equation. We can state this symbolically as

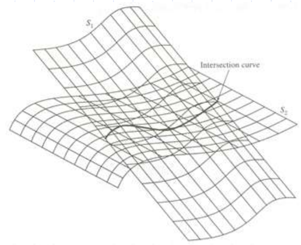
  P[u(t), w(t)] = points on the curve, on the patch
- We can find an object-space tangent vector $P^t{}_t$ to the curve at any point t on it by first computing $c^t{}_t$, the tangent vector to the curve at u(t), w(t) in the u, w plane. Determine the direction cosines of this vector (that is, with respect of the u, w coordinate system), and denote them k_u and k_w. Now compute $P^u{}_{uw}$ and $P^w{}_{uw}$ at the u, w point determined by c(t) at t. We find that

$$p^t{}_t = k_u \frac{p^u{}_{uw}}{\left|p^u{}_{uw}\right|} + k_w \frac{p^w{}_{uw}}{\left|p^w{}_{uw}\right|}$$

- It is the direction of $P^t{}_t$ that interests us; the magnitude has no particular meaning.

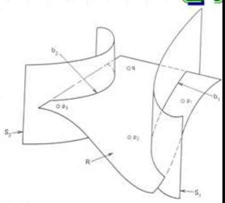
surface-to-surface intersection

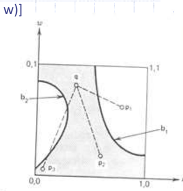
- The problem of surface-to-surface intersection is defined by the equation
- P(u, v) – P(t, w) = 0
- where the P(u, v) = 0 and P(t, w) = 0 are the equations of the two surfaces.

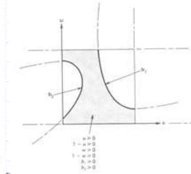


- Compute the intersections as sets of u, w points, then apply curve fitting techniques to define these curves or a set of composite curves.
- To obtain the boundary curve in object space, compute points on the u, w coordinate plane from $u_i(t)$ and $w_i(t)$.
- To obtain (x, y, z) points on the curve, substitute the values of u and w into the appropriate surface equation.

p = [x(u, w) y(u, w) z(u, w)]

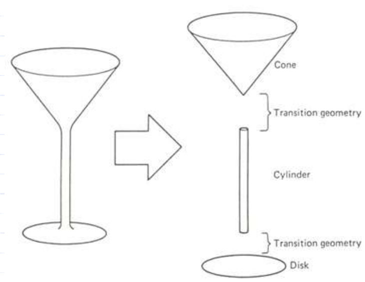






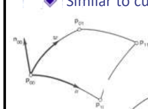
Surfaces with irregular boundaries

- In general, the intersection curves do not coincide with the iso-parametric curves on the surface, are called irregular boundaries of a surface.
- Segment the total surface so that each region is itself mathematically tractable where as the total surface may not be mathematically tractable.

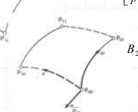


Reparametrization/ Subdivision

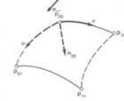
- Reversal of direction of one or both of the parametric variables
- Similar to curves



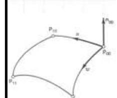
$$B_1 = \begin{bmatrix} p_{00} & p_{10} & p^w{}_{00} & p^w{}_{01} \\ p_{10} & p_{11} & p^w{}_{10} & p^w{}_{11} \\ p^u{}_{00} & p^u{}_{01} & p^{uw}{}_{00} & p^{uw}{}_{01} \\ p^u{}_{10} & p^u{}_{11} & p^{uw}{}_{10} & p^{uw}{}_{11} \end{bmatrix}$$



$$B_2 = \begin{bmatrix} q_{00} = p_{00} & q_{01} = p_{11} & q^w{}_{00} = p^w{}_{10} & q^w{}_{01} = p^w{}_{11} \\ q_{10} = p_{00} & q_{11} = p_{01} & q^w{}_{10} = p^w{}_{00} & q^w{}_{11} = p^w{}_{01} \\ q^u{}_{00} = -p_{10} & q^u{}_{01} = -p^u{}_{11} & q^{uw}{}_{00} = -p^{uw}{}_{10} & q^{uw}{}_{01} = -p^{uw}{}_{11} \\ q^u{}_{10} = -p^u{}_{00} & q^u{}_{11} = -p^u{}_{01} & q^{uw}{}_{10} = -p^{uw}{}_{00} & q^{uw}{}_{11} = -p^{uw}{}_{01} \end{bmatrix}$$



$$B_3 = \begin{bmatrix} r_{00} = p_{01} & r_{01} = p_{00} & r^w{}_{00} = -p^w{}_{01} & r^w{}_{01} = -p^w{}_{00} \\ r_{10} = p_{11} & r_{11} = p_{10} & r^w{}_{10} = -p^w{}_{11} & r^w{}_{11} = -p^w{}_{10} \\ r^u{}_{00} = p^u{}_{01} & r^u{}_{01} = p^u{}_{00} & r^{uw}{}_{00} = -p^{uw}{}_{01} & r^{uw}{}_{01} = -p^{uw}{}_{00} \\ r^u{}_{10} = p^u{}_{11} & r^u{}_{11} = p^u{}_{10} & r^{uw}{}_{10} = -p^{uw}{}_{11} & r^{uw}{}_{11} = -p^{uw}{}_{10} \end{bmatrix}$$

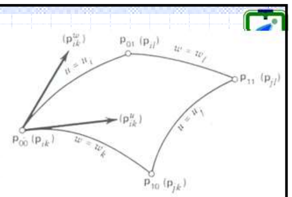


$$B_4 = \begin{bmatrix} s_{00} = p_{11} & s_{01} = p_{10} & s^w{}_{00} = -p^w{}_{11} & s^w{}_{01} = -p^w{}_{10} \\ s_{10} = p_{01} & s_{11} = p_{00} & s^w{}_{10} = -p^w{}_{0w} & s^w{}_{11} = -p^w{}_{00} \\ s^u{}_{00} = -p^u{}_{11} & s^u{}_{01} = -p^u{}_{10} & s^{uw}{}_{00} = p^{uw}{}_{11} & s^{uw}{}_{01} = p^{uw}{}_{10} \\ s^u{}_{10} = -p^u{}_{01} & s^u{}_{11} = -p^u{}_{00} & s^{uw}{}_{10} = p^{uw}{}_{01} & s^{uw}{}_{11} = p^{uw}{}_{00} \end{bmatrix}$$

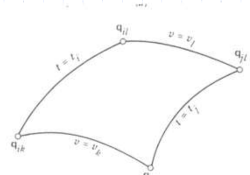
Reparametrization

- Varying parameter range

$$B_1 = \begin{bmatrix} p_{ik} & p_{il} & p^w{}_{ik} & p^w{}_{il} \\ p_{jk} & p_{jl} & p^w{}_{jk} & p^w{}_{jl} \\ p^u{}_{ik} & p^u{}_{il} & p^{uw}{}_{ik} & p^{uw}{}_{il} \\ p^u{}_{jk} & p^u{}_{jl} & p^{uw}{}_{jk} & p^{uw}{}_{jl} \end{bmatrix}$$



$$B_2 = \begin{bmatrix} q_{ik} & q_{il} & q^v{}_{ik} & q^v{}_{il} \\ q_{jk} & q_{jl} & q^v{}_{jk} & q^v{}_{jl} \\ q^u{}_{ik} & q^t{}_{il} & q^{tv}{}_{ik} & q^{tv}{}_{il} \\ q^u{}_{jk} & q^t{}_{jl} & q^{tv}{}_{jk} & q^{tv}{}_{jl} \end{bmatrix}$$



The relationship between the elements of B1 and B2 are the same as obtained in curves.

- The corner points are related directly.

$$q_{ik} = p_{ik}$$
$$q_{jk} = p_{jk}$$
$$q_{il} = p_{il}$$
$$q_{jl} = p_{jl}$$
$$q^t = \frac{u_j - u_i}{t_j - t_i} p^u$$
$$q^r = \frac{w_t - w_k}{v_t - v_k} p^w$$

- For the cross-derivatives, we obtain

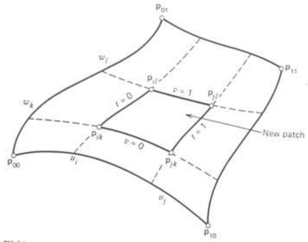
$$q^{tv} = \frac{(u_j - u_i)(w_t - w_k)}{(t_j - t_i)(v_t - v_k)} p^{uw}$$

- If uj and ui are successive pairs of integers, then $u_j - u_i = 1$; similarly for $t_j - t_i$, $w_t - w_k$, and $v_t - v_k$.

Subdivision of a surface

- Given a patch whose geometric coefficients are denoted B_1,
- find the matrix B_2 of a new patch that is a sub-patch of the given patch and bounded by curves u_i, u_j, w_k and w_l.
- The corner points of the new patch
- where the p vectors and q vectors are elements of B_1 and B_2 respectively.

$$q_{00} = p_{ik}$$
$$q_{10} = p_{jk}$$
$$q_{01} = p_{il}$$
$$q_{11} = p_{jl}$$



October 17, 2024 Dr. Prashant K. Jain (IIITDMJ) 147

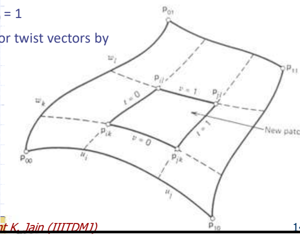
Subdivision of a surface

- The tangent vectors are

$$q^t{}_{00} = (u_j - u_i)p^u{}_{ik} \qquad q^v{}_{00} = (w_l - w_k)p^w{}_{ik}$$
$$q^t{}_{10} = (u_j - u_i)p^u{}_{jk} \qquad q^v{}_{10} = (w_l - w_k)p^w{}_{jk}$$
$$q^t{}_{01} = (u_j - u_j)p^u{}_{il} \qquad q^v{}_{01} = (w_l - w_k)p^w{}_{il}$$
$$q^t{}_{11} = (u_j - u_j)p^u{}_{jl} \qquad q^v{}_{11} = (w_l - w_k)p^w{}_{jl}$$

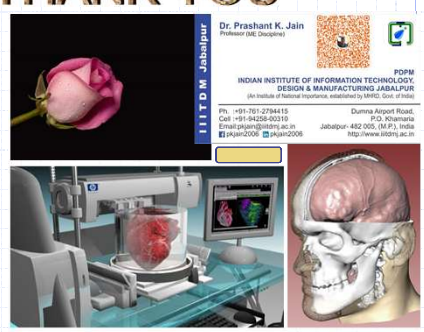
- Remember that $t_1 - t_0 = 1$ and $v_1 - v_0 = 1$
- We can obtain the cross-derivatives or twist vectors by evaluating $p^{uw}{}_{ik}$, $p^{uw}{}_{jk}$, $p^{uw}{}_{il}$ and $p^{uw}{}_{jl}$.

$$q^{tv}{}_{00} = (u_j - u_i)(w_l - w_k)p^{uw}{}_{ik}$$
$$q^{tv}{}_{10} = (u_j - u_i)(w_l - w_k)p^{uw}{}_{jk}$$
$$q^{tv}{}_{01} = (u_j - u_i)(w_l - w_k)p^{uw}{}_{il}$$
$$q^{tv}{}_{11} = (u_j - u_i)(w_l - w_k)p^{uw}{}_{jl}$$



October 17, 2024 Dr. Prashant K. Jain (IIITDMJ) 148

THANK YOU



October 17, 2024 Dr. Prashant K. Jain (IIITDMJ) 149

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October 17, 2024 Dr. Prashant K. Jain (IIITDMJ) 150