Computer Aided Geometric Design

Solids

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(An Institute of National Importance (INI) established by MHRD, Govt. of INDIA)

Solid Modeling

Hyperpatch Representation of Free-from Solids

- Hermite Hyperpatch
  - Algebraic form
  - Geometric form
  - n-point form
- Bezier Hyperpatch (l by m by n points)
- B-spline Hyperpatch (l by m by n points)
- NURBS Hyperpatch (l by m by n points & weights)

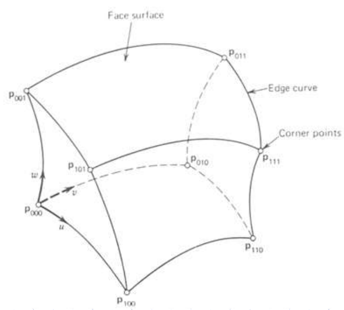
Hyperpatch

A hyperpatch is a patch-bounded collection of points whose coordinates are given by continuous, three-parameter, single values mathematical functions of the form

$x = x(u, v, w)$,

$y = y(u, v, w)$

$z = z(u, v, w)$



Rectangular parametric solid

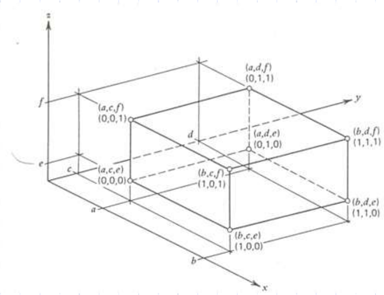
$x = (b - a)u + a$

$y = (d - c)v + c$

$z = (f - e)w + e$

where

$u, v, w \in [0, 1]$



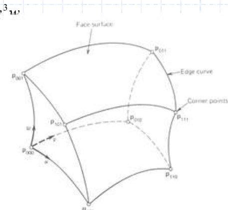
Algebraic and Geometric Form

The algebraic form of a tricubic solid is given by the following polynomial equation:

$$p(u,v,w) = \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} a_{ijk} u^i v^j w^k \qquad u,v,w \in [0,1]$$

The a_{ijk} vectors are called the algebraic coefficients of the solid.

$$p(u,v,w) = a_{333}u^3v^3w^3 + a_{332}u^3v^3w^2 + a_{331}u^3v^3w + a_{323}u^3v^2w^3 + \cdots \\ \cdot \\ \cdot \\ + a_{000}$$



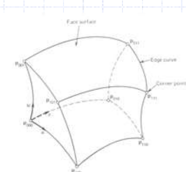
We can take the same approach with the geometric form, with the result

$$p = F_i(u)F_j(v)F_k(w)b_{ijk}$$

The F terms are the familiar blending functions F_1, F_2, F_3 and F_4, as determined by the subscripted index.

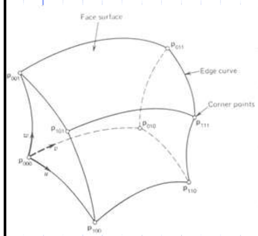
At each of the eight corners, we find the following boudary conditions:

The corner point itself, three tangent vectors, three twist vectors and a vector defined by the third-order mixed partial derivatives of the function.



There are eight vectors defining boundary conditions at each of the eight corners, resulting in a total of 64 vectors.

At the corner defined by p(1, 0, 1) eight boundary condition vectors are as:



$$p_{101} \quad \text{corner point}$$

$$\left.\begin{matrix} p^{u}_{101} \\ p^{v}_{101} \\ p^{w}_{101} \end{matrix}\right\} \quad \text{tangent vectors} \quad \frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}$$

$$\left.\begin{matrix} p^{uv}_{101} \\ p^{uw}_{101} \\ p^{vw}_{101} \end{matrix}\right\} \quad \text{twist vectors} \quad \frac{\partial^2}{\partial u \partial v}, \frac{\partial^2}{\partial u \partial w}, \frac{\partial^2}{\partial v \partial w}$$

$$p^{uvw}_{101} \quad \text{triple mixed partial} \quad \frac{\partial^3}{\partial u \partial v \partial w}$$

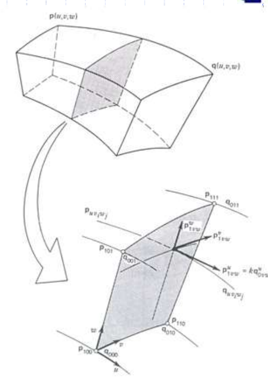
Continuity and composite solids

- To ensure C_0 continuity, we must meet the condition on the common surface:

$$p_{1vw} = q_{0vw}$$

- For C_1 continuity, the tangent vectors of the curves at these points must be colinear; thus

$$p^{u}_{1vw} = kq^{u}_{0vw}$$



Solid Models

Internal representation vs external representation

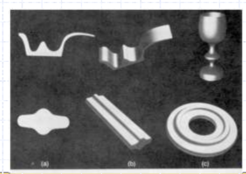
Internal representation is how computer stores the model.

- Sweeping - translation, rotation, or arbitrary sweep
- Pure Primitive Instancing - Parametric model
- Constructive Solid Geometry (CSG) - solid primitives combined with Boolean operators
- Boundary Representation (B-Rep) - bounding face, edge, and vertices
- Decomposition Methods
  - Voxel Based Modeling : model of fixed size spatial cells
  - Octree Based Modeling: model of variable spatial cells

Solid Modeling

Sweep Representation of Solids

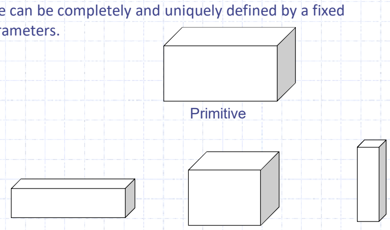
- A polygon or a surface or a solid is swept along a path of given curve to obtain a solid.
- Linear Sweep (Extruded Solid) and Circular Sweep (Solid of Revolution) are special cases of Generic Sweep.
- This is a convenient method of construction and storage of solids.



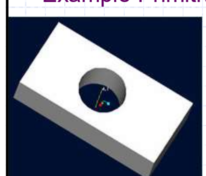
Solid Modeling

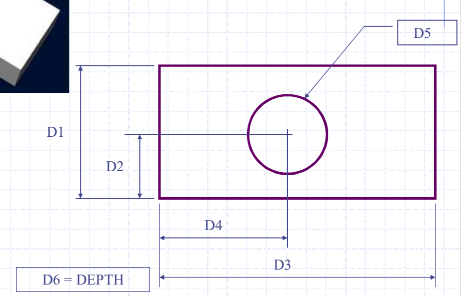
Primitive Instancing

- Part is modeled as a combination of primitives.
- Large number of primitives need to be stored in library to model variety of objects.
- Every primitive can be completely and uniquely defined by a fixed number of parameters.



Example Primitive

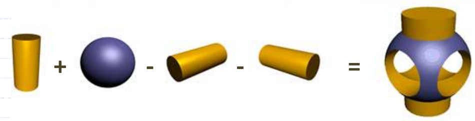




Solid Modeling

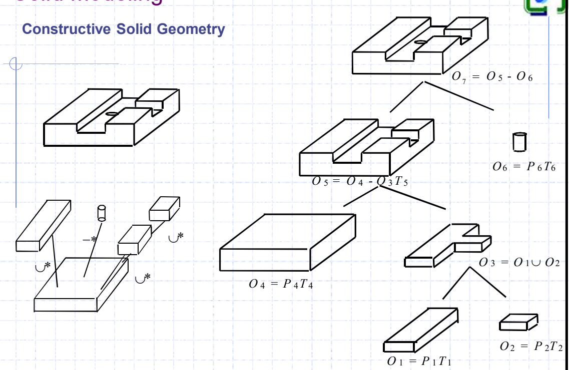
Constructive Solid Geometry (CSG)

- This is a solid modeling method that combines simple solid primitives to build more complex models using Boolean operators: *union, difference* and *intersection.*
- The resulting model is a procedural model stored in the mathematical form of a binary tree where leaf nodes are solid primitives, correctly sized and positioned, and each branch node is a Boolean operator.
- Many objects can be built using simple primitives such as cuboids, cylinder, cone, sphere, torus etc.
- Solids obtained by sweeping can be used together with primitives.
- Convenient method for construction and storage of solid objects.



Solid Modeling

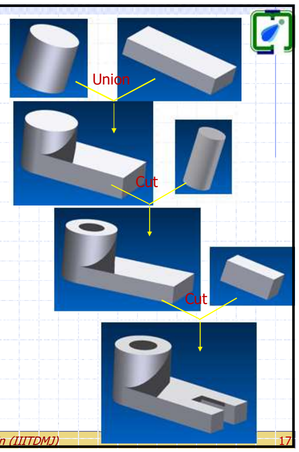
Constructive Solid Geometry

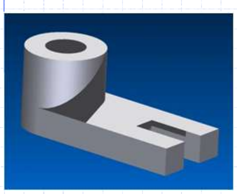


A CSG Tree

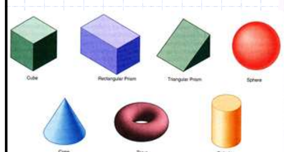
Solid Modeling

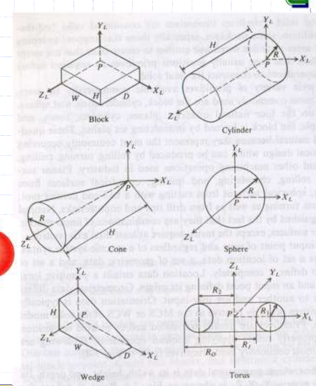
Constructive Solid Geometry



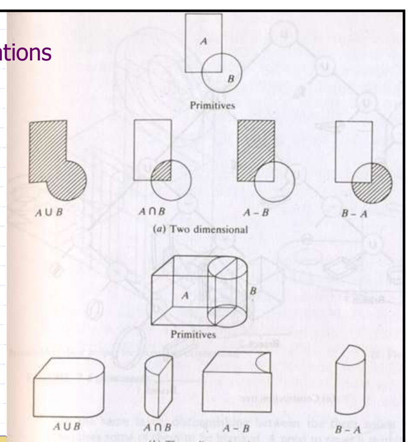


Some CSG Primitives

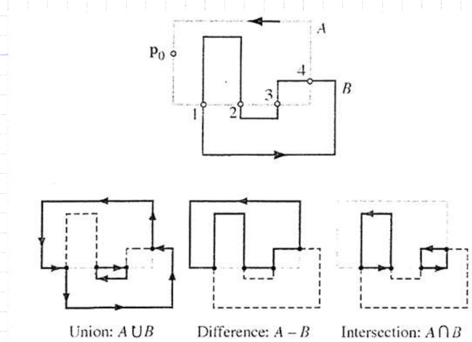




Boolean Operations



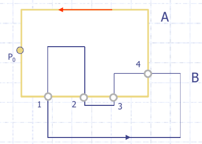
Boolean Operations



Algorithm for finding unions of two polygons

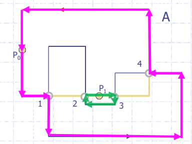
begin by combining two polygons A and B as shown in Figure. It is required to find union of two polygons A and B

1. Find intersections of edges of A and B as shown in figure
2. Segment edges of A and B. Thus if boundary of polygon A is parameterized from $u = 0$ to $u = 1$ and B from $v = 0$ to $v = 1$, then the boundary of A in the example has four segments: $u \in [u_1 \quad u_2]$, $u \in [u_2 \quad u_3]$, $u \in [u_3 \quad u_4]$, $u \in [u_4 \quad u_1]$. B also has four segments $v \in [v_4 \quad v_3]$, $v \in [v_3 \quad v_2]$, $v \in [v_2 \quad v_1]$, $v \in [v_1 \quad v_4]$,
3. Select a point on the boundary of polygon A that is outside B, say P_0. Then that segment is also outside B.



Algorithm for finding unions of two polygons

4. Starting at P_0, trace the boundary of A to the next intersection point with B, i.e. point 1.
5. Find the intersecting segment of B, and trace it along the direction of increasing v to its intersection with A, point 4. At this point, we discover that we have traced back to the starting segment on A, but we have not yet exhausted the list of segments. We have found only one loop.
6. Find a point on one of the remaining segments of A that is outside of B say point P_1. Then that segment too, is outside B.
7. Trace this boundary segment to the next intersection point with B, i.e. point 3.
8. Repeat steps 5, tracing B to point 2. We have completed another loop.
9. Because there are no more qualifying segments to trace, union operation is complete. Notice second loop is parameterized in clockwise direction it is a hole.



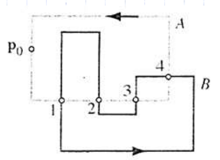
Algorithm for difference and intersection

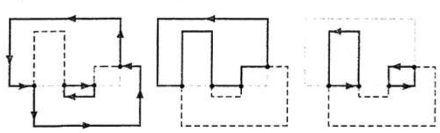
Difference operation

The algorithm is same as union operation, except trace the boundary of polygon B clockwise.

Subtraction operation

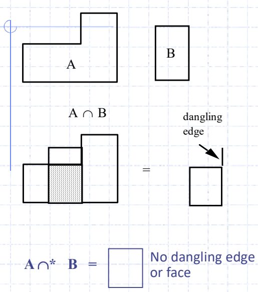
The algorithm is similar to union operation, except segment tracing must start from a point on a boundary of A that is inside of B.

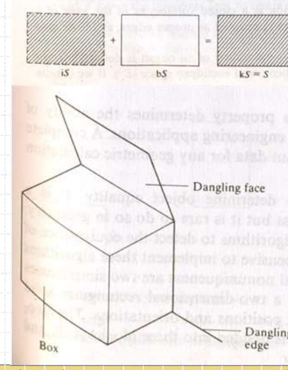




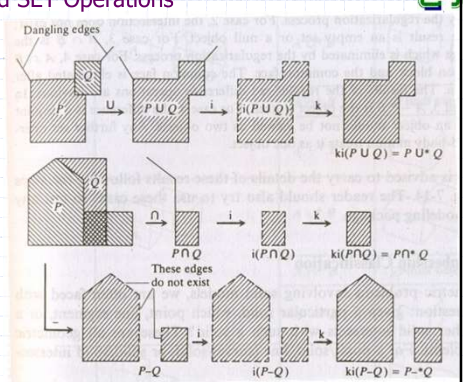
Union: $A \cup B$ Difference: $A - B$ Intersection: $A \cap B$

Regularized SET Operations

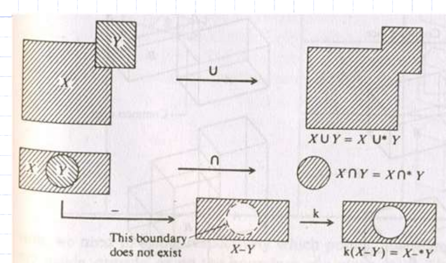




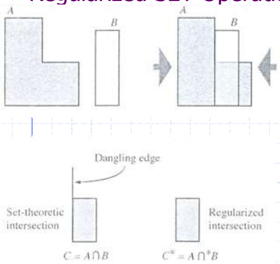
Regularized SET Operations

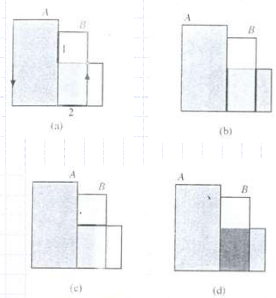


Regularized SET Operations



Regularized SET Operations



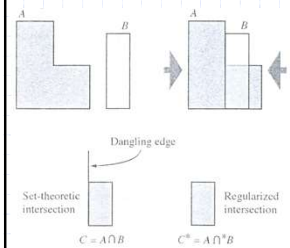


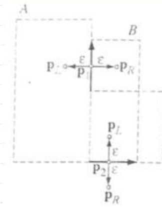
$C = A \cap B$

$C = (bA \cup iA) \cap (bB \cup iB)$

$C = (bA \cap bB) \cup (iA \cap bB) \cup (bA \cap iB) \cup (iA \cap iB)$

Regularized SET Operations



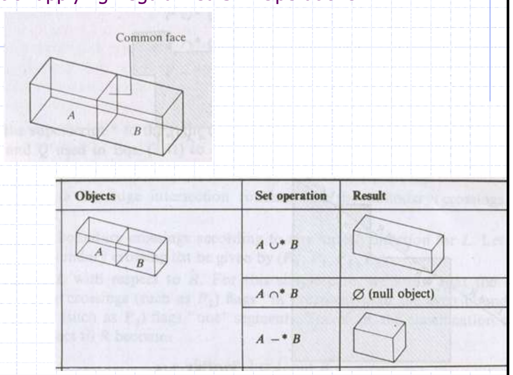


Segment 1	In A	In B
P_R	0	1
P_L	1	0

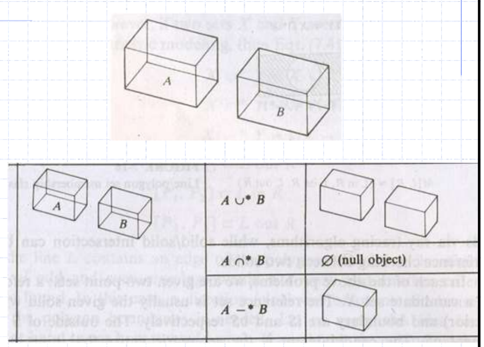
Segment 2	In A	In B
P_R	0	0
P_L	1	1

Note: 1 = Yes, 2 = No

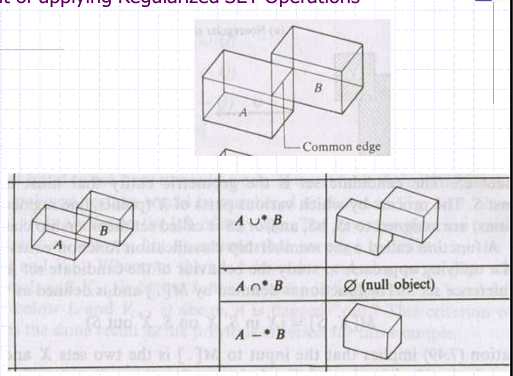
Result of applying Regularized SET Operations



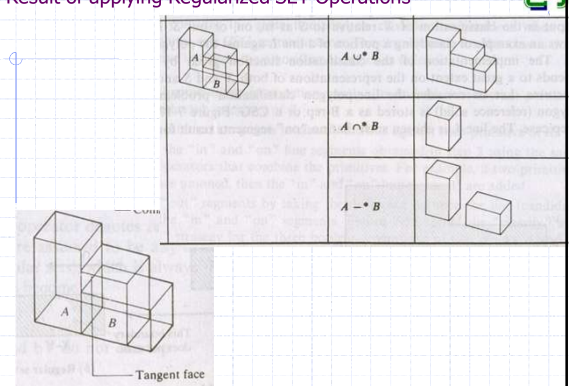
Result of applying Regularized SET Operations



Result of applying Regularized SET Operations



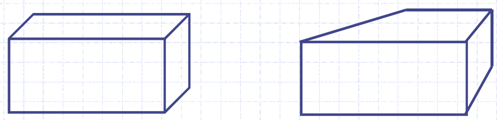
Result of applying Regularized SET Operations



Solid Modeling

Boundary Representation (B-Rep)

- In this representation solid objects are represented as unions of their boundaries or enclosing surfaces. The enclosing surfaces can include planar polygons, quadrics and free-form surface patches.
- In this scheme topological and geometric information are explicitly defined.
- Topological information provides the relationships among vertices, edges/curves and faces/patches. In addition to connectivity, topological information also includes orientation of edges and faces etc.
- Geometric information usually consists of equations of the edges/curves and faces/patches.



Objects with same topology but with different geometry

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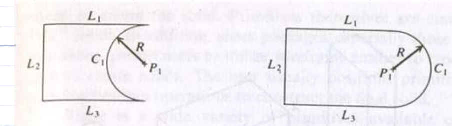
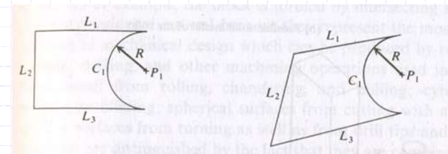
Topology and geometry



Same geometry but different topology



Same topology but different geometry

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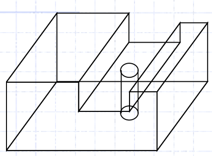
Wireframe Model



Modeling with all edges of an object.

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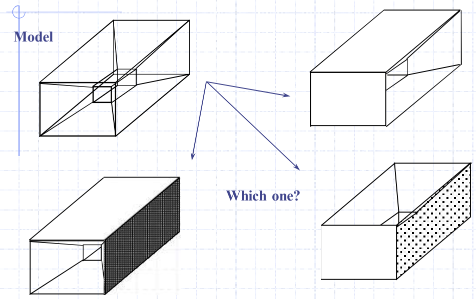
Ambiguity of Wireframe Model



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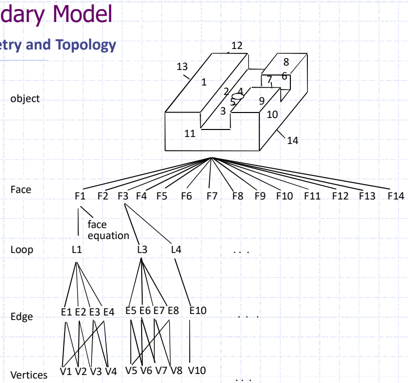
Boundary Model

Geometry and Topology



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Boundary Representations

Validity of A B-REP Model

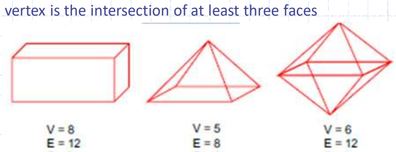
Euler's formula

$$V - E + F = 2$$

V: Vertices, E: Edges, F: Faces

Conditions

- No hole is present
- The solid is connected
- Each edge is adjacent to exactly two faces and terminated by two vertices
- Each vertex is the intersection of at least three faces



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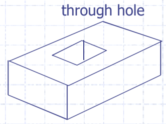
Boundary Representations

- Euler's formula can be generalized to a polyhedron with holes and multiple components

$$V - E + F - L = 2\,(B - G)$$

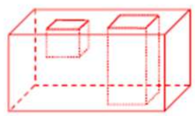
Where:
L: Number of inner loops (connected edges),
B: Number of exterior shells (connected surfaces),
G: Number of passage ways (through hole)

through hole



$V - E + F - L = 2\,(B - G)$

$16 - 24 + 10 - 2 = 2\,(1 - 1)$



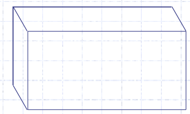
$V - E + F - L = 2\,(B - G)$

$24 - 36 + 15 - 3 = 2\,(1 - 1)$

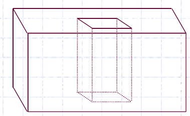
Example: Euler's formula

$$V - E + F - L = 2\,(B - G)$$

Consider sample parts:



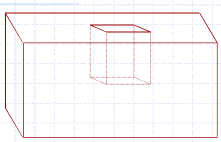
E = 12,
V = 8,
F = 6



E = 24
V = 16,
F = 10 (6 plus additional 4)
L = 2 (as its through hole)
B = 1
G = 1

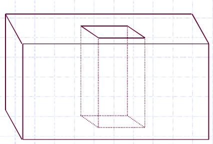
Example: Part with blind hole

$$V - E + F - L = 2\,(B - G)$$



E = 24
F = 6+5 = 11
V = 16, L = 1
B = 1
G = 0

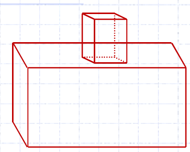
If this part contained a blind hole, then?



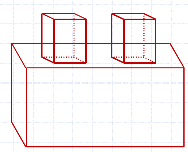
E = 24
V = 16,
F = 10 (6 plus additional 4)
L = 2 (as its through hole)
B = 1
G = 1

Example: Part with Projection

$$V - E + F - L = 2\,(B - G)$$



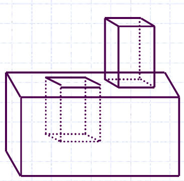
For 1 projections on a part,
E = 24,
V = 16,
F =11(6 + 4 +1)
G = 0
L = 1 (at base of projection)



For 2 projections on a part,
E=36,
V=24,
F=16,
L=2 (at base of projection)
G=0

Example: Projection and Blind Hole

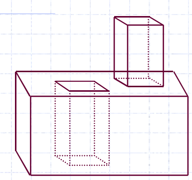
$$V - E + F - L = 2\,(B - G)$$



E=12+24 (from prev. slide) = 36
V= 8+16 = 24
F= 5+11 = 16
L=1+1 (at base of projection and top of hole)

Example: Projection and Through Hole

$$V - E + F - L = 2\,(B - G)$$



E = 12+24 (from prev. slide) = 36
V = 8+16=24
F = 4+11 =15
L = 1+2 (at base of projection, top and bottom of hole)
B = 1
G = 1

Solid Modeling

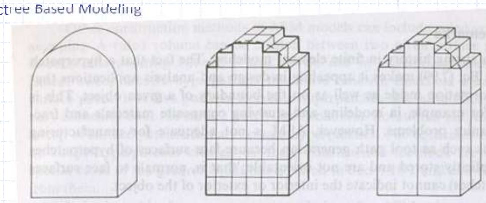
Boundary Representation (B-Rep)

- This representation is a convenient for evaluation of many intrinsic and relational properties of solids.
- Conversion from of Constructive Solid Geometry (CSG) to Boundary Representation (Brep) is called *boundary evaluation.*

Solid Modeling

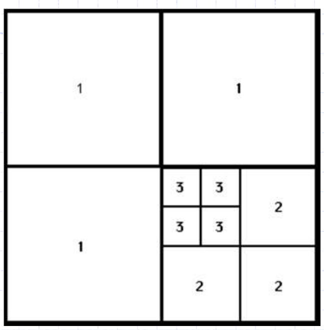
Decomposition Methods

- In decomposition methods, a solid is decomposed into a collection of adjoining, non-intersecting solid primitives.
- Two of the important methods under this category are:
  - Voxel Based Modeling
  - Octree Based Modeling

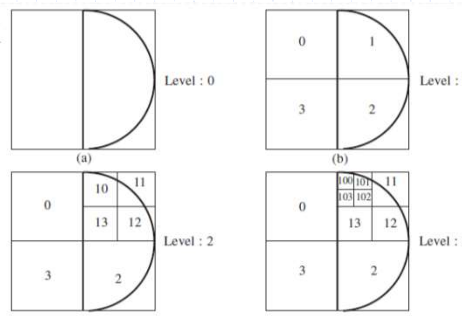


(a) Solid (b) Spatial occupancy enumeration (c) Octree encoding

Decomposition Methods

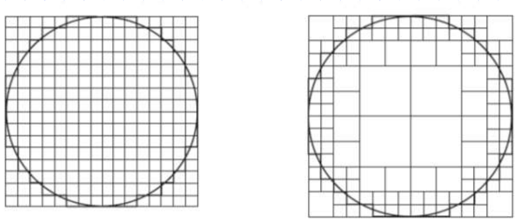


Quadtree decomposition



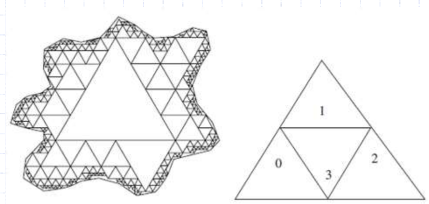
(a) (b) (c) (d)

Quadtree decomposition



(a) (b)

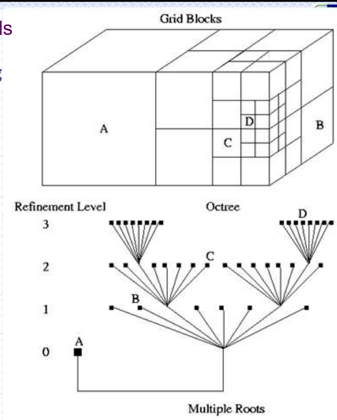
Quadtree decomposition with equilateral triangles



(a) (b)

Decomposition Methods

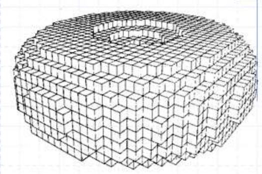
Octree Based Modeling

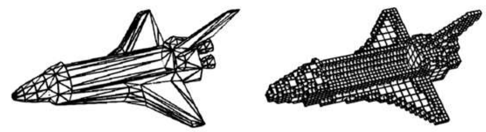


Decomposition Methods

Voxel Based Modeling

- **Pros:**
  - Easy to verify if a point (a voxel) is inside or outside an object
  - Boolean operations are easy to apply
- **Cons:**
  - Memory costs are high
  - Resolution is limited to size and shape of voxel





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THANK YOU