MID-Semester Examination 2024

ME 601: Computer Aided Geometric Design

Duration: 120 Min. Maximum Marks: 60

Name: ______________________ Roll Number: ________________

All questions are compulsory.

1. Explain with suitable examples, use and need of the homogeneous coordinate system. 05
   - Rotation, scaling, shearing, reflection are in form of matrix multiplication, but the translation takes the form of vector addition. This makes it inconvenient to concatenate transformation involving translation. It is desirable to express all geometric transformations in the form of matrix multiplication on representing points by their homogeneous coordinates.
   - Provides an effective way to unify the description of geometric transformations as matrix multiplication.
   - In homogeneous coordinates, an n-dimensional space is mapped into (n+1) dimensional space that is a point P(x,y,z) has homogeneous coordinates (x',y',z',h) where h is any scalar factor which is not equal to 0.
   - A point in homogeneous coordinates (x, y, z, h), h≠0, corresponds to the 3-D vertex (x/h, y/h, z/h) in Cartesian coordinates.
   - Homogeneous coordinates in 3D give rise to 4 dimensional position vector

2. Discuss various properties and limitations of Bezier curves. 10
   1. The basis functions are real.
   2. The degree of the polynomial defining the curve segment is one less than the number of defining polygon points.
   3. The curve generally follows the shape of the defining polygon.
   4. The first and last points on the curve are coincident with the first and last points of the defining polygon.
   5. The tangent vectors at the ends of the curve have the same direction as the first and last polygon spans, respectively.
   6. The curve is contained within the convex hull of the defining polygon, i.e., within the largest convex polygon defined by the polygon vertices.
   7. The curve exhibits the variation diminishing property. Basically this means that the curve does not oscillate about any straight line more often than the defining polygon.
   8. The curve is invariant under an affine transformation.
   - With increase in no. of vertices degree of polynomial increases
   - Global nature

3. In case of axonometric projection derive the expression and find out the angle of rotation with respect to two principal axes to keep foreshortening ratio equal in all three directions. Also find out the foreshortening factor. 15

Specific rotation angle can be obtained as:

Resulting Transformation is : T = Ry Rx Pz

$$[T]=\begin{bmatrix}\cos\phi & 0 & -\sin\phi & 0\\ 0 & 1 & 0 & 0\\ \sin\phi & 0 & \cos\phi & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & \sin\theta & 0\\ 0 & -\sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$[T]=\begin{bmatrix}\cos\phi & \sin\phi\sin\theta & 0 & 0\\ 0 & \cos\theta & 0 & 0\\ \sin\phi & -\cos\phi\sin\theta & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Unit vectors on the x, y and z principal axes transform to

$$[U^*]=U\,T=\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}\cos\phi & \sin\phi\sin\theta & 0 & 0\\ 0 & \cos\theta & 0 & 0\\ \sin\phi & -\cos\phi\sin\theta & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$[U^*]=\begin{bmatrix}\cos\phi & \sin\phi\sin\theta & 0 & 1\\ 0 & \cos\theta & 0 & 1\\ \sin\phi & -\cos\phi\sin\theta & 0 & 1\end{bmatrix}$$

$$f_x=\sqrt{x_x^{\;2}+y_x^{\;2}}=\sqrt{\cos^2\phi+\sin^2\phi\sin^2\theta} \qquad \text{---------------------(A)}$$

$$f_y=\sqrt{x_y^{\;2}+y_y^{\;2}}=\sqrt{\cos^2\theta} \qquad \text{---------------------(B)}$$

$$f_z=\sqrt{x_z^{\;2}+y_z^{\;2}}=\sqrt{\sin^2\phi+\cos^2\phi\sin^2\theta} \qquad \text{---------------------(C)}$$

Equating equation A and B

$$\cos^2\phi+\sin^2\phi\sin^2\theta=\cos^2\theta$$

$$1-\sin^2\phi+\sin^2\phi\sin^2\theta=1-\sin^2\theta$$

$$\sin^2\phi(1-\sin^2\theta)=\sin^2\theta$$

$$\sin^2\phi=\frac{\sin^2\theta}{1-\sin^2\theta} \qquad \text{-----------------------------(D)}$$

We know

$$\cos^2\phi=1-\sin^2\phi$$

$$\cos^2\theta=1-\sin^2\theta$$

Now equate equation B and C:

$$\cos^2\theta=\sin^2\phi+\cos^2\phi\sin^2\theta$$

$$1-\sin^2\theta=\sin^2\phi+(1-\sin^2\phi)\sin^2\theta$$

We know

$$\cos^2\phi=1-\sin^2\phi$$

$$\cos^2\theta=1-\sin^2\theta$$

$$1-\sin^2\theta = \sin^2\phi + \sin^2\theta - \sin^2\phi\sin^2\theta$$

$$1-2\sin^2\theta = \sin^2\phi\left(1-\sin^2\theta\right)$$

$$\sin^2\phi = \frac{1-2\sin^2\theta}{1-\sin^2\theta} \qquad \text{------------------------------(E)}$$

Now Equating equation D and E

$$\frac{\sin^2\theta}{1-\sin^2\theta} = \frac{1-2\sin^2\theta}{1-\sin^2\theta}$$

Gives $\sin^2\theta = {}^1\!/_3$, $\sin\theta = \pm\, {}^1\!/_{\sqrt{3}}$ $\qquad \theta = \pm 35.26^0$

Using equation D

$$\sin^2\phi = \frac{{}^1\!/_3}{1-{}^1\!/_3} = {}^1\!/_2 \qquad \phi = \pm 45^0$$

Then using equation B $\qquad f_z = \sqrt{\cos^2\theta} = \sqrt{{}^2\!/_3} = 0.8165$

4. Mathematically prove following with respect to a cubic Bezier curve: 15
    a. First and last point of the Bezier curve is same as the first and last point of the defining control polygon of the Bezier curve.
    b. Tangent vector for a Bezier curve at the start and end point has the same direction as the first and last polygon span.
    c. Second derivative or curvature of Bezier curve at start and end point depends on three nearest polygon points or two nearest polygon spans.

The derivatives of the basis function are obtained by formally differentiating equation $J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$. Specifically, to find maximum value of blending function

$$J'_{n,i}(t) = \binom{n}{i}\left\{i\, t^{i-1}(1-t)^{n-i} - (n-i)t^i(1-t)^{n-i-1}\right\} = 0$$

$$i\, t^{i-1}(1-t)^{n-i} - (n-i)t^i(1-t)^{n-i-1} = 0$$

$$i\, t^{i-1}(1-t)^{n-i} = (n-i)t^i(1-t)^{n-i-1}$$

$$\frac{i\, t^i}{t}(1-t)^{n-i} = (n-i)t^i\frac{(1-t)^{n-i}}{(1-t)}$$

$$\frac{i}{t} = \frac{(n-i)}{(1-t)}$$

$$\frac{(1-t)}{t} = \frac{(n-i)}{i}$$

$$\frac{1}{t} - 1 = \frac{n}{i} - 1 \quad === \quad \frac{1}{t} = \frac{n}{i}$$

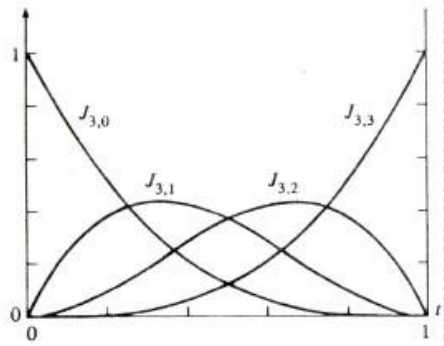
For example, each of the four blending functions shown in Figure for n=3 is a cubic. The maximum value of each blending function occurs at *t =i/n*.

$$J_{n,i}(t) = \binom{n}{i} \frac{i^i (n-i)^{n-i}}{n^n}$$

For example, for a cubic n=3, The maximum values for $J_{3,1}$ and $J_{3,2}$ occur at *1/3* and *2/3*, respectively, with values

$$J_{3,1}\left(\frac{1}{3}\right) = \frac{4}{9}$$

$$J_{3,2}\left(\frac{2}{3}\right) = \frac{4}{9}$$



(b)

Examining eqns. for the first point on the curve i.e. at t=0 shows that

$$J_{n,0}(0) = \frac{n!\,(0)^i(1-0)^{n-0}}{n!\,(n-i)!} = 1 \qquad i = 0$$

$$J_{n,i}(0) = \frac{n!\,(0)^i(1-0)^{n-i}}{n!\,(n-i)!} = 0 \qquad i \neq 0$$

Thus $\mathrm{P}(0) = B_0 J_{n,0}(0) = B_0$

This shows that the first point on the Bezier curve and on its defining polygon are coincident .

Similarly for the last point on the curve, i.e. at t=1

$$J_{n,n}(1) = \frac{n!\,(1)^i(0)^{n-n}}{n!\,(1)} = 1 \qquad i = n$$

$$J_{n,i}(1) = \frac{n!\,(1)^i(1-1)^{n-i}}{n!\,(n-i)!} = 0 \qquad i \neq n$$

Thus $\mathrm{P}(1) = B_n J_{n,n}(1) = B_n$

This shows that the last point on the Bezier curve and the last point on its defining polygon are coincident.

Example on Derivatives

Consider the four points Bezier polygon

$$P(t) = B_0 J_{3,0}(t) + B_1 J_{3,1}(t) + B_2 J_{3,2}(t) + B_3 J_{3,3}(t)$$

Hence first derivative is

$$P'(t) = B_0 J'_{3,0}(t) + B_1 J'_{3,1}(t) + B_2 J'_{3,2}(t) + B_3 J'_{3,3}(t)$$

Second derivative is

$$P''(t) = B_0 J''_{3,0}(t) + B_1 J''_{3,1}(t) + B_2 J''_{3,2}(t) + B_3 J''_{3,3}(t)$$

Differentiating the basis functions directly yields

$$J_{3,0}(t) = t^0(1-t)^3 \rightarrow J'_{3,0}(t) = -3(1-t)^2 \rightarrow J''_{3,0}(t) = 6(1-t)$$

$$J_{3,1}(t) = 3t(1-t)^2 \rightarrow J'_{3,1}(t) = 3(1-t)^2 - 6t(1-t) \rightarrow J''_{3,1}(t) = -6(2-3t)$$

$$J_{3,2}(t) = 3t^2(1-t) \rightarrow J'_{3,2}(t) = 6t(1-t) - 3(t)^2 \rightarrow J''_{3,2}(t) = 6(1-3t)$$

$$J_{3,3}(t) = t^3 \rightarrow J'_{3,3}(t) = 3(t)^2 \rightarrow J''_{3,3}(t) = 6t$$

Example on Derivatives

Evaluating the results at t=0 yields

$$J'_{3,0}(0) = -3, \quad J'_{3,1}(0) = 3, \quad J'_{3,2}(0) = 0, \quad J'_{3,3}(0) = 0$$

Substituting yields

$$P'(0) = -3B_0 + 3B_1 = 3(-B_0 + B_1)$$

Thus the direction of the tangent vector at the beginning of the curve is the same as that of the first polygon span

At the end of the curve, t=1 and

$$J'_{3,0}(1) = 0, \quad J'_{3,1}(1) = 0, \quad J'_{3,2}(1) = -3, \quad J'_{3,3}(1) = 3$$

Substituting yields

$$P'(1) = -3B_2 + 3B_3 = 3(-B_2 + B_3)$$

Thus, the direction of the tangent vector at the end of the curve is the same as that of the last polygon span.

Example on Derivatives

Evaluating the results at t=0 yields

$$J''_{3,0}(0) = 6,\ J''_{3,1}(0) = -12,\ J''_{3,2}(0) = 6,\ J''_{3,3}(0) = 0$$

Substituting yields

$$P''(0) = 6B_0 - 12B_1 + 6B_2 = 6(B_0 - 2B_1 + B_2)$$

Thus second derivative or curvature of Bezier curve at start point depends on three nearest polygon points or two nearest polygon spans.

At the end of the curve, t=1 and

$$J'_{3,0}(1) = 0,\ J''_{3,1}(1) = 6,\ J''_{3,2}(1) = -12,\ J''_{3,3}(1) = 6$$

Substituting yields

$$P''(1) = 6B_1 - 12B_2 + 6B_3 = 6(B_1 - 2B_2 + B_3)$$

Thus second derivative or curvature of Bezier curve at end point depends on three nearest polygon points or two nearest polygon spans.

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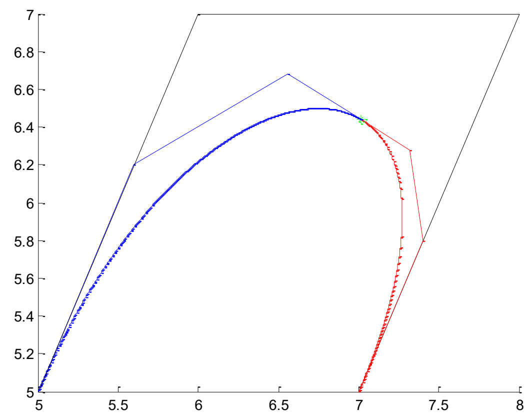
5. Define a Bezier curve with four polygon vertices B_0[5 5 4], B_1[6 7 4], B_2[8 7 4] and B_3[7 5 4], split this curve into two curves each one being a cubic Bezier curve and find out the control points of these two curves using continuity conditions if the original curve is split at point corresponding to parametric value u = 0.6. 15



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